Online Appendix – Second-Best Income Taxation and Education Policy with Endogenous Human Capital and Borrowing Constraints

Bas Jacobs**
Erasmus University Rotterdam, Tinbergen Institute, and CESifo

Hongyan Yang
Universität Bremen and ZeS

November 11, 2014

A Second-order conditions

We derive first the second-order conditions of the household's maximization problem when credit constraints are slack and then when credit constraints are binding.

A.1 Slack credit constraint

We employ a two-step budgeting procedure to derive the second-order conditions. We assume linear homogenous sub-utility in consumption u over c^1 and c^2 and we define the real price-index for consumption p such that $p \equiv \frac{c^2_{n\omega}+(1+r)c^1_{n\omega}}{u(c^1_{n\omega},c^2_{n\omega})}$. Due to homogeneity the consumption price p is independent of n and ω. In the remainder of this section we drop the indices n and ω. Using the budget constraint

$$(1+r)c^1 + c^2 = (1-t)n\phi(e)l + (1+r)(-(1-s)\,e + \omega + g) + g, \tag{1}$$

we can rewrite the individual maximization problem as an unconstrained maximization problem:

$$\max_{\{e,l\}} (1-t)n\phi(e)l + (1+r)(-(1-s)\,e + \omega + g) + g - pv(l). \tag{2}$$

The first-order conditions are given by

$$(1-t)n\phi'(e)l - (1+r)\,(1-s) = 0, \tag{3}$$

$$(1-t)n\phi(e) - v'(l)p = 0. \tag{4}$$

Hence, the Hessian matrix with second-order derivatives is

$$H \equiv \begin{bmatrix} (1-t)n\phi''l & (1-t)n\phi' \\ (1-t)n\phi' & -v''p \end{bmatrix}. \tag{5}$$

The first principal minor, $(1-t)n\phi'' l$, is negative, because by assumption $\phi'' < 0$. Therefore, for the Hessian to be negative semi-definite, the second principal minor should be positive:

$$-(1-t)n\phi'' l v'' p - \left((1-t)n\phi'\right)^2 > 0. \tag{6}$$

By defining the elasticity of labor supply as $\varepsilon \equiv \left(\frac{v''(l)l}{v'(l)}\right)^{-1}$, and using the first-order condition for labor supply as well as the assumed human capital production function $\phi(e) = e^{\beta}$, we can rewrite the above inequality as

$$\beta(1+\varepsilon) < 1. \tag{7}$$

The second-order condition thus requires that the elasticity of labor supply and the elasticity of human capital production function should be sufficiently small so as to avoid corner solutions. In the second stage of the budgeting procedure, individuals maximize $u(c^1, c^2)$ subject to the constraint $pu(c^1, c^2) = (1+r)c^1 + c^2$. The associated second-order condition $u_{11}u_{22} - u_{12}^2 \geq 0$ is always satisfied since u is assumed to be strictly concave.

A.2 Binding credit constraint

With a binding credit constraint, savings are zero ($a = 0$). Hence, we can again obtain an unconstrained maximization problem upon substitution of budget constraints in the utility function:

$$\max_{\{e,l\}} u(-(1-s)e + \omega + g, (1-t)n\phi(e)l + g) - v(l). \tag{8}$$

The first-order conditions are given by

$$-(1-s)u_1(\cdot) + u_2(\cdot)(1-t)n\phi'(e)l = 0, \tag{9}$$

$$u_2(\cdot)(1-t)n\phi(e) - v'(l) = 0. \tag{10}$$

The Hessian matrix H with second-order partial derivatives is given by

$$H \equiv \begin{bmatrix} \begin{matrix}(1-s)^2 u_{11} - 2u_{12}(1-t)(1-s)n\phi' l \\ +u_{22}\left((1-t)n\phi' l\right)^2 + u_2(1-t)n\phi'' l\end{matrix} & \begin{matrix}-u_{12}(1-s)(1-t)n\phi \\ +u_{22}(1-t)^2 n^2 \phi\phi' l + u_2(1-t)n\phi'\end{matrix} \\ \begin{matrix}-u_{12}(1-t)(1-s)n\phi \\ +u_{22}(1-t)^2 n^2 \phi\phi' l + u_2(1-t)n\phi'\end{matrix} & u_{22}\left((1-t)n\phi\right)^2 - v'' \end{bmatrix}. \tag{11}$$

For the Hessian matrix to be negative semi-definite, the principal minors should switch signs. The first principal minor

$$(1-s)^2 u_{11} - 2u_{12}(1-t)(1-s)n\phi' l + u_{22}\left((1-t)n\phi' l\right)^2 + u_2(1-t)n\phi'' l < 0, \tag{12}$$

is negative since all terms of (12) are negative under the assumptions that the consumption utility function is concave in both arguments ($u_{11} < 0$, $u_{22} < 0$), the human capital production function is concave ($\phi'' < 0$), and consumption in two periods are complementary ($u_{12} \geq 0$).

The second principal minor should therefore be positive:

$$\begin{aligned}&\left((1-s)^2 u_{11} - 2u_{12}(1-t)(1-s) n\phi' l + u_{22}\left((1-t)n\phi' l\right)^2 + u_2(1-t)n\phi'' l\right) \\ &\times \left(u_{22}\left((1-t)n\phi\right)^2 - v''\right) \\ &- \left(-u_{12}(1-t)(1-s) n\phi + u_{22}(1-t)^2 n^2 \phi\phi' l + u_2(1-t)n\phi'\right)^2 > 0.\end{aligned} \tag{13}$$

Use the first-order conditions (9) and (10), the definition $\varepsilon \equiv \left(\frac{v''(l)l}{v'(l)}\right)^{-1}$ and the assumption that $\phi(e) = e^\beta$ to reformulate the above inequality as

$$\begin{aligned}&u_{11}u_{22} + u_{22}u_1 \frac{(\beta-1)}{(1-s)e} - \frac{u_2}{\varepsilon}\frac{\beta}{(1-s)e}\left(u_{11}\frac{u_2}{u_1} - 2u_{12} + u_{22}\frac{u_1}{u_2} + u_2\frac{(\beta-1)}{(1-s)e}\right) \\ &> u_{12}^2 + \left(u_2\frac{\beta}{(1-s)}\right)^2 - 2u_{12}u_2\frac{\beta}{(1-s)e} + 2u_{22}u_1\frac{\beta}{(1-s)e}.\end{aligned} \tag{14}$$

In deriving the second-order conditions, we assume that the utility function is linear homogenous and we use the properties $u_{11}c_1 = -u_{12}c_2$ and $u_{12}c_1 = -u_{22}c_2$ to find:

$$u_{11}u_{22} - u_{12}^2 = 0. \tag{15}$$

Using (15) we can rewrite (14) as

$$\beta(1+\varepsilon) - 1 < -\frac{u_{22}u_1(1-s)e}{u_2^2}\left(1 + \frac{(1-\beta)\varepsilon}{\beta} + 2\varepsilon\right) - \frac{u_{11}(1-s)e}{u_1} + 2\frac{u_{12}(1-s)e}{u_2}(1+\varepsilon). \tag{16}$$

Because $u_{22} < 0$, $u_{11} < 0$ and $\beta - 1 < 0$, the right-hand-side of equation (16) is always positive. Consequently,

$$\beta(1+\varepsilon) < 1 \tag{17}$$

is sufficient for (16) to be fulfilled. Therefore, $\beta(1+\varepsilon) < 1$ is the sufficient second-order condition for the households' maximization problem for both the cases of slack and binding credit constraints.

B Derivation Slutsky equations

In calculating the optimal-tax formulae we employ the Slutsky equations. We derive the Slutsky equations where a uniform income compensation is given in *both* periods, that is, by a higher lump-sum transfer g, so as to keep utility constant. Totally differentiating utility and the household budget constraints in both periods gives:

$$\mathrm{d}U_{n\omega} = u_{1,n\omega}\mathrm{d}c^1_{n\omega} + u_{2,n\omega}\mathrm{d}c^2_{n\omega} - v_{l,n\omega}\mathrm{d}l_{n\omega} = 0, \tag{18}$$

$$\mathrm{d}c^1_{n\omega} = -(1-s)\mathrm{d}e_{n\omega} + \mathrm{d}\omega + \mathrm{d}g - \mathrm{d}a_{n\omega} + e_{n\omega}\mathrm{d}s, \tag{19}$$

$$\mathrm{d}c^2_{n\omega} = (1-t)nl_{n\omega}\phi'_{n\omega}\mathrm{d}e_{n\omega} + (1-t)n\phi_{n\omega}\mathrm{d}l_{n\omega} - nl_{n\omega}\phi_{n\omega}\mathrm{d}t + (1+r)\mathrm{d}a_{n\omega} + \mathrm{d}g. \tag{20}$$

Substitution of (19) and (20) in (18) yields:

$$\mathrm{d}U = \underbrace{\left(u_{2,n\omega}(1-t)nl_{n\omega}\phi'_{n\omega} - (1-s)u_{1,n\omega}\right)}_{=0}\mathrm{d}e_{n\omega} + u_{1,n\omega}\mathrm{d}\omega + u_{1,n\omega}\mathrm{d}g + \underbrace{\left(u_{2,n\omega}(1+r) - u_{1,n\omega}\right)\mathrm{d}a_{n\omega}}_{=0} \tag{21}$$

$$+ u_{1,n\omega}e\mathrm{d}s + \underbrace{\left(u_{2,n\omega}(1-t)n\phi_{n\omega} - v_{l,n\omega}\right)}_{=0}\mathrm{d}l_{n\omega} - u_{2,n\omega}nl_{n\omega}\phi_{n\omega}\mathrm{d}t + u_{2,n\omega}\mathrm{d}g$$

$$= u_{2,n\omega}nl_{n\omega}\phi_{n\omega}\mathrm{d}t + u_{1,n\omega}\mathrm{d}\omega + (u_{1,n\omega} + u_{2,n\omega})\mathrm{d}g + u_{1,n\omega}e_{n\omega}\mathrm{d}s = 0.$$

By the envelope theorem the terms $\mathrm{d}e_{n\omega}$ and $\mathrm{d}l_{n\omega}$ drop out via the first-order conditions for labor supply (eq. 10 in paper) and education (eq. 9 in paper). The term for $da_{n\omega}$ drops out both in the case where credit constraints are binding ($da_{n\omega} = 0$) and where it is slack, via first-order condition (eq. 7 in paper). If the utility-compensation is given in both periods by a higher transfer g, the Slutsky equations are thus given by:

$$\frac{\partial l_{n\omega}}{\partial t} = \frac{\partial l^c_{n\omega}}{\partial t} - \frac{u_{2,n\omega}}{u_{1,n\omega} + u_{2,n\omega}}nl_{n\omega}\phi_{n\omega}\frac{\partial l_{n\omega}}{\partial g}, \tag{22}$$

$$\frac{\partial e_{n\omega}}{\partial t} = \frac{\partial e^c_{n\omega}}{\partial t} - \frac{u_{2,n\omega}}{u_{1,n\omega} + u_{2,n\omega}}nl_{n\omega}\phi_{n\omega}\frac{\partial e_{n\omega}}{\partial g}, \tag{23}$$

$$\frac{\partial l_{n\omega}}{\partial s} = \frac{\partial l^c_{n\omega}}{\partial s} + \frac{u_{1,n\omega}}{u_{1,n\omega} + u_{2,n\omega}}e_{n\omega}\frac{\partial l_{n\omega}}{\partial g}, \tag{24}$$

$$\frac{\partial e_{n\omega}}{\partial s} = \frac{\partial e^c_{n\omega}}{\partial s} + \frac{u_{1,n\omega}}{u_{1,n\omega} + u_{2,n\omega}}e_{n\omega}\frac{\partial e_{n\omega}}{\partial g}. \tag{25}$$

C Compensated elasticities

This appendix derives the compensated elasticities in our model. First, we derive some properties of the assumed homothetic utility function, which will be exploited later on. Second, we derive the elasticities when capital markets are perfect. Third, we derive the elasticities when capital markets are imperfect. We log-linearize the model around an initial equilibrium and we use a tilde to denote relative changes of variables, e.g. $\tilde{x} \equiv \mathrm{d}x/x$, except for the tax and subsidy rates, whose relative changes are expressed as $\tilde{t} \equiv \mathrm{d}t/(1-t)$ and $\tilde{s} \equiv \mathrm{d}s/(1-s)$. In the remainder of this section on the elasticities we will suppress the subscripts n and ω.

C.1 Homothetic utility function

Throughout we assume that sub-utility over consumption $u(\cdot)$ is homothetic and is a monotonic transformation Ω of a linear homogeneous function $\mathcal{U}$:

$$u(c^1, c^2) = \Omega\left[\mathcal{U}(c^1, c^2)\right], \quad \Omega' > 0. \tag{26}$$

$\mathcal{U}$ has the standard properties that it is homogeneous of degree one and its first derivatives are homogeneous of degree zero:

$$\mathcal{U}_1 c^1 + \mathcal{U}_2 c^2 = \mathcal{U}, \quad \mathcal{U}_{11} c^1 + \mathcal{U}_{12} c^2 = 0, \quad \mathcal{U}_{21} c^1 + \mathcal{U}_{22} c^2 = 0. \tag{27}$$

We define the share of first-period consumption ϑ, the intertemporal elasticity of substitution in consumption σ, and the elasticity ρ of Ω' as:

$$\vartheta \equiv \frac{\mathcal{U}_1 c^1}{\mathcal{U}}, \quad \sigma \equiv \left(\frac{\mathcal{U}_{21}\mathcal{U}}{\mathcal{U}_1 \mathcal{U}_2}\right)^{-1}, \quad \rho \equiv \frac{\Omega'' \mathcal{U}}{\Omega'}. \tag{28}$$

We log-linearize the marginal utilities u_1 and u_2 and using (27), (28) we find:

$$\tilde{u}_1 = \left(-\frac{(1-\vartheta)}{\sigma} + \rho\vartheta\right)\tilde{c}^1 + \left(\frac{(1-\vartheta)}{\sigma} + \rho(1-\vartheta)\right)\tilde{c}^2, \tag{29}$$

$$\tilde{u}_2 = \left(\frac{\vartheta}{\sigma} + \rho\vartheta\right)\tilde{c}^1 + \left(-\frac{\vartheta}{\sigma} + \rho(1-\vartheta)\right)\tilde{c}^2. \tag{30}$$

And, from this follows that:

$$\tilde{u}_1 - \tilde{u}_2 = \frac{1}{\sigma}(\tilde{c}^2 - \tilde{c}^1). \tag{31}$$

C.2 Elasticities perfect capital markets

This subsection derives the compensated elasticities when capital markets are perfect. In particular, we log-linearize the following equations around an initial equilibrium:

$$\bar{U} = u(c^1, c^2) - v(l), \tag{32}$$

$$u_2(1-t)n\phi(e) = v'(l), \tag{33}$$

$$\frac{(1-t)nl\phi'(e)}{1-s} = 1 + r, \tag{34}$$

$$\frac{u_1}{u_2} = 1 + r. \tag{35}$$

First, log-linearizing the utility function (32) at constant utility (i.e. $\mathrm{d}\bar{U} = 0$) yields:

$$\frac{c^1 u_1}{c^2 u_2}\tilde{c}^1 + \tilde{c}^2 = \frac{(1-t)n\phi l}{c^2}\tilde{l}. \tag{36}$$

Substitute the first-order condition for labor supply (33) to find the linearized utility function:

$$\vartheta\tilde{c}^1 + (1-\vartheta)\tilde{c}^2 = \frac{(1-\vartheta)}{\mu}\tilde{l}, \tag{37}$$

where $\mu \equiv \frac{c^2}{(1-t)n\phi l} > 0$. Second, the first-order condition for labor supply (33) is linearized to find:

$$\tilde{l} = \varepsilon\beta\tilde{e} + \varepsilon\left(\frac{\vartheta}{\sigma} + \rho\vartheta\right)\tilde{c}^1 + \varepsilon\left(-\frac{\vartheta}{\sigma} + \rho(1-\vartheta)\right)\tilde{c}^2 - \varepsilon\tilde{t}, \tag{38}$$

where $\varepsilon \equiv (lv''/v')^{-1} > 0$. Third, linearizing the first-order condition for human capital investment (34) yields:

$$(1-\beta)\tilde{e} = \tilde{l} - \tilde{t} + \tilde{s}, \tag{39}$$

where $1-\beta \equiv -e\phi''/\phi' > 0$. Fourth, linearizing the first-order condition for consumption (35) gives:

$$\tilde{u}_1 - \tilde{u}_2 = \frac{1}{\sigma}(\tilde{c}^2 - \tilde{c}^1) = 0. \tag{40}$$

Equations (37), (38), (39), and (40) form a set of 4 equations in the relative changes of the 4 endogenous variables, which can be solved analytically. From (40) follows that $\tilde{c}^2 = \tilde{c}^1$ so that the relative change in education and labor supply can be solved as functions of the relative changes in policy parameters:

$$\tilde{l} = -\left(\frac{\mathcal{E}(\beta+\alpha)}{(1-\beta)-\mathcal{E}\beta}\right)\tilde{t} + \left(\frac{\mathcal{E}\beta}{(1-\beta)-\mathcal{E}\beta}\right)\tilde{s}, \tag{41}$$

$$\tilde{e} = -\left(\frac{1+\mathcal{E}}{(1-\beta)-\mathcal{E}\beta}\right)\tilde{t} + \left(\frac{1}{(1-\beta)-\mathcal{E}\beta}\right)\tilde{s}, \tag{42}$$

where $\mathcal{E} \equiv \varepsilon\left[1+\varepsilon(1-\rho)(1-\omega)/\mu\right]^{-1}$ is the compensated labor-supply elasticity with respect to the wage rate. Consequently, we find the following elasticities that enter the optimal-tax expressions:

$$\varepsilon_{lt} = \frac{\mathcal{E}}{1-\beta(1+\mathcal{E})} > 0, \quad \varepsilon_{ls} = \frac{\mathcal{E}\beta}{1-\beta(1+\mathcal{E})} > 0, \tag{43}$$

$$\varepsilon_{et} = \frac{1+\mathcal{E}}{1-\beta(1+\mathcal{E})} > 0, \quad \varepsilon_{es} = \frac{1}{1-\beta(1+\mathcal{E})} > 0. \tag{44}$$

C.3 Imperfect capital markets

The system of equations to be linearized with imperfect capital markets is different. In particular, it is now given by

$$\bar{U} = u(c^1, c^2) - v(l), \tag{45}$$

$$u_2(1-t)n\phi(e) = v'(l), \tag{46}$$

$$\frac{(1-t)nl\phi'(e)}{1-s} = \frac{u_1}{u_2}, \tag{47}$$

$$a + a_o = \omega + g + a_o - (1-s)e - c^1 = 0. \tag{48}$$

Note that the last equation is the credit constraint, which needs to be satisfied in this case. First, since the utility constraint (45) and the first-order condition for labor supply (46) are the same as with perfect capital markets, log-linearizing the utility function (45) at constant utility (i.e. $\mathrm{d}\bar{U} = 0$) and first-order condition for labor supply (46) yields the same results:

$$\vartheta\tilde{c}^1 + (1-\vartheta)\tilde{c}^2 = \frac{(1-\vartheta)}{\mu}\tilde{l}, \tag{49}$$

$$\tilde{l} = \varepsilon\beta\tilde{e} + \varepsilon\left(\frac{\vartheta}{\sigma} + \rho\vartheta\right)\tilde{c}^1 + \varepsilon\left(-\frac{\vartheta}{\sigma} + \rho(1-\vartheta)\right)\tilde{c}^2 - \varepsilon\tilde{t}. \tag{50}$$

Second, linearizing the first-order condition for human capital (47) yields:

$$\tilde{c}^2 = \tilde{c}^1 + \sigma(\tilde{s} - \tilde{t} + \tilde{l} - (1-\beta)\tilde{e}). \tag{51}$$

Third, linearizing the credit constraint (48) (at $\mathrm{d}a = 0$) gives:

$$\delta_e(\tilde{s} - \tilde{e}) + \delta_g \tilde{g} = \tilde{c}^1, \tag{52}$$

where $\delta_e \equiv (1-s)e/c^1$ and $\delta_g \equiv g/c_1$ are share parameters of educational investment $(1-s)e$ and transfers g in first-period net income $c^1 = \omega + g + a_o - (1-s)e$. Therefore, equations (49), (50), (51), and (52) form a system of 4 equations in the relative changes of the 4 unknowns. We will solve the model only for the special case in which utility $u(\cdot)$ is linearly homogeneous, i.e. $\rho = 0$. The relative changes in education and labor supply can be written as functions of the relative changes in the policy parameters:

$$\tilde{l} = \left[\frac{\psi\theta\varepsilon(1-\vartheta) + \psi\left(\vartheta - \frac{1}{\vartheta}\right)\sigma\varepsilon(\beta + \vartheta(1-\beta))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right]\tilde{t} + \left[\frac{\psi\theta\varepsilon\vartheta + \psi\left(\delta_e - \vartheta\sigma\right)\varepsilon(\beta + \vartheta(1-\beta))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right]\tilde{s} \tag{53}$$
$$+\left[\frac{\varepsilon(\beta + \vartheta(1-\beta))\psi\delta_g}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right]\tilde{g},$$

$$\begin{aligned}\tilde{e} = {} & \frac{(1+\varepsilon\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}\left[\frac{\psi\theta\varepsilon(1-\vartheta) + \psi\left(\vartheta - \frac{1}{\vartheta}\right)\sigma\varepsilon(\beta + \vartheta(1-\beta))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right]\tilde{t} + \frac{\varepsilon(1-\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}\tilde{t} \\ & + \frac{(1+\varepsilon\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}\left[\frac{(\psi\theta\varepsilon\vartheta + \psi\left(\delta_e - \vartheta\sigma\right)\varepsilon(\beta + \vartheta(1-\beta)))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right]\tilde{s} + \frac{\varepsilon\vartheta}{\varepsilon(\beta + \vartheta(1-\beta))}\tilde{s} \\ & + \frac{(1+\varepsilon\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}\left[\frac{\varepsilon(\beta + \vartheta(1-\beta))\psi\delta_g}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right]\tilde{g}.\end{aligned} \tag{54}$$

where $\psi \equiv \left(\frac{(1-\vartheta)}{\mu} + \frac{\sigma}{\varepsilon\vartheta} + \vartheta\sigma\right)^{-1} > 0$ and $\theta \equiv \frac{\sigma\beta}{\vartheta} - \vartheta\sigma(1-\beta) - \delta_e$. Therefore, the elasticities of this model are given by:

$$-\varepsilon_{lt} = \frac{\psi\theta\varepsilon(1-\vartheta) + \psi\left(\vartheta - \frac{1}{\vartheta}\right)\sigma\varepsilon(\beta + \vartheta(1-\beta))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}, \tag{55}$$

$$\varepsilon_{ls} = \frac{(\psi\theta\varepsilon\vartheta + \psi\left(\delta_e - \vartheta\sigma\right)\varepsilon(\beta + \vartheta(1-\beta)))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}, \tag{56}$$

$$-\varepsilon_{et} = \frac{(1+\varepsilon\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}\left[\frac{\psi\theta\varepsilon(1-\vartheta) + \psi\left(\vartheta - \frac{1}{\vartheta}\right)\sigma\varepsilon(\beta + \vartheta(1-\beta))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right] + \frac{\varepsilon(1-\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}, \tag{57}$$

$$\varepsilon_{es} = \frac{(1+\varepsilon\vartheta)}{\varepsilon(\beta + \vartheta(1-\beta))}\left[\frac{(\psi\theta\varepsilon\vartheta + \psi\left(\delta_e - \vartheta\sigma\right)\varepsilon(\beta + \vartheta(1-\beta)))}{(\varepsilon(\beta + \vartheta(1-\beta)) - \psi\theta(1+\varepsilon\vartheta))}\right] + \frac{\varepsilon\vartheta}{\varepsilon(\beta + \vartheta(1-\beta))}. \tag{58}$$

Clearly, under credit constraints we cannot establish that $\varepsilon_{ls} = \beta\varepsilon_{lt}$. Consequently, the efficiency result of Bovenberg and Jacobs (2005) cannot be found with credit constraints.