Online Appendix – Pigou Meets Mirrlees: On the Irrelevance of Tax Distortions for the Second-Best Pigouvian Tax

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October 16, 2014

1 Model

Individuals maximize utility:

$$u_n \equiv u(c_n, q_n, l_n, E), \quad u_c, u_q, -u_l, u_E > 0, \quad u_{cc}, u_{ll}, u_{qq}, u_{EE} < 0, \quad \forall n. \tag{1}$$

subject to their budget constraints:

$$c_n + (1+\tau)q_n = (1-t)nl_n + T, \quad \forall n. \tag{2}$$

The Lagrangian for utility maximization is

$$\mathcal{L} \equiv u(c_n, q_n, l_n, E) + \lambda_n \left[(1-t)nl_n + T - c_n - (1+\tau)q_n\right], \quad \forall n, \tag{3}$$

where λ_n is the Lagrange multiplier on the household budget constraint and it denotes the marginal utility of income.

Households take environmental quality E as given when deciding on their consumption plans. First-order conditions are:[1]

$$\frac{\partial \mathcal{L}}{\partial c_n} = u_c(\cdot) - \lambda_n = 0, \quad \forall n, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial q_n} = u_q(\cdot) - \lambda_n(1+\tau) = 0, \quad \forall n \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial l_n} = u_l(\cdot) + \lambda_n(1-t)n = 0, \quad \forall n, \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_n} = (1-t)nl_n + T - c_n - (1+\tau)q_n = 0, \quad \forall n. \tag{7}$$

From these equations follow equations (3) and (4) in the text:

$$\frac{-u_l}{u_c} = (1-t)n, \quad \forall n, \tag{8}$$

$$\frac{u_q}{u_c} = 1+\tau, \quad \forall n. \tag{9}$$

The indirect utility function is designated by $v_n \equiv v(T, t, \tau, E) \equiv u(\hat{c}_n, \hat{q}_n, \hat{l}_n, E)$, $\forall n$, where hats denote optimized values of each commodity and labor supply. Application of Roy's identity produces the following derivatives of the indirect utility function:

$$\frac{\partial v_n}{\partial T} = \lambda_n, \quad \forall n, \tag{10}$$

$$\frac{\partial v_n}{\partial t} = -\lambda_n n l_n, \quad \forall n, \tag{11}$$

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[1] Strict quasi-concavity of the utility function ensures that second-order conditions for a maximum are fulfilled under linear instruments. When non-linear instruments are employed, additional single-crossing and monotonocity conditions need to be satisfied too. See the later derivations using non-linear taxes.

$$\frac{\partial v_n}{\partial \tau} = -\lambda_n q_n, \quad \forall n, \tag{12}$$

$$\frac{\partial v_n}{\partial E} = \lambda_n \frac{u_E}{u_c}, \quad \forall n. \tag{13}$$

Environmental quality (E) is a linear function of aggregate consumption of dirty goods:

$$E \equiv E_0 - \alpha N \int_{\mathcal{N}} q_n \mathrm{d}F(n), \quad E_0,\ \alpha > 0, \tag{14}$$

The government maximizes a Bergson-Samuelson social welfare function, which is a concave sum of individual utilities:

$$N \int_{\mathcal{N}} \Psi(u_n) \mathrm{d}F(n), \quad \Psi'(u_n) > 0, \quad \Psi''(u_n) \leq 0. \tag{15}$$

subject to the government budget constraint:

$$N \int_{\mathcal{N}} (tnl_n + \tau q_n) \mathrm{d}F(n) = NT + R. \tag{16}$$

2 Optimal linear taxation

The Lagrangian for maximizing social welfare is given by (where the whole expression has been divided by the population size N to save on notation):

$$\begin{aligned} \max_{\{T,t,\tau,E\}} \mathcal{L} \equiv & \int_{\mathcal{N}} \Psi(v(T,t,\tau,E)) \mathrm{d}F(n) \\ & + \eta \left(\int_{\mathcal{N}} (tnl_n + \tau q_n) \mathrm{d}F(n) - T - \frac{R}{N} \right) - \mu \left(\frac{E - E_0}{N} + \alpha \int_{\mathcal{N}} q_n \mathrm{d}F(n) \right). \end{aligned} \tag{17}$$

The Lagrange multiplier η denotes the marginal social value of public resources and the Lagrange multiplier μ denotes the marginal social cost per capita (measured in social welfare units) of providing a better environmental quality E.

The first-order conditions for an optimal allocation are given by:

$$\frac{\partial \mathcal{L}}{\partial T} = \int_{\mathcal{N}} \left[\Psi' \lambda_n - \eta + \eta t n \frac{\partial l_n}{\partial T} + (\eta \tau - \alpha \mu) \frac{\partial q_n}{\partial T} \right] \mathrm{d}F(n) = 0, \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial t} = \int_{\mathcal{N}} \left[-nl_n \Psi' \lambda_n + \eta n l_n + \eta t n \frac{\partial l_n}{\partial t} + (\eta \tau - \alpha \mu) \frac{\partial q_n}{\partial t} \right] \mathrm{d}F(n) = 0, \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \tau} = \int_{\mathcal{N}} \left[-q_n \Psi' \lambda_n + \eta q_n + \eta t n \frac{\partial l_n}{\partial \tau} + (\eta \tau - \alpha \mu) \frac{\partial q_n}{\partial \tau} \right] \mathrm{d}F(n) = 0, \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial E} = \int_{\mathcal{N}} \left[\frac{u_E}{u_c} \Psi' \lambda_n - \frac{\mu}{N} + \eta t n \frac{\partial l_n}{\partial E} + (\eta \tau - \alpha \mu) \frac{\partial q_n}{\partial E} \right] \mathrm{d}F(n) = 0, \tag{21}$$

where the derivatives of the indirect utility function (10) – (13) are used in each subsequent expression.[2]

The optimal income tax, the optimal environmental tax, and the optimal provision of environmental quality are derived by employing the Slutsky equations for labor supply, the demand for the dirty commodity, and the demand for environmental quality:

$$\frac{\partial l_n}{\partial t} = \frac{\partial l_n^*}{\partial t} - nl_n \frac{\partial l_n}{\partial T}, \quad \forall n, \tag{22}$$

$$\frac{\partial q_n}{\partial t} = \frac{\partial q_n^*}{\partial t} - nl_n \frac{\partial q_n}{\partial T}, \quad \forall n, \tag{23}$$

$$\frac{\partial l_n}{\partial \tau} = \frac{\partial l_n^*}{\partial \tau} - q_n \frac{\partial l_n}{\partial T}, \quad \forall n, \tag{24}$$

$$\frac{\partial q_n}{\partial \tau} = \frac{\partial q_n^*}{\partial \tau} - q_n \frac{\partial q_n}{\partial T}, \quad \forall n, \tag{25}$$

$$\frac{\partial l_n}{\partial E} = \frac{\partial l_n^*}{\partial E} + \frac{u_E}{u_c} \frac{\partial l_n}{\partial T}, \quad \forall n, \tag{26}$$

$$\frac{\partial q_n}{\partial E} = \frac{\partial q_n^*}{\partial E} + \frac{u_E}{u_c} \frac{\partial q_n}{\partial T}, \quad \forall n. \tag{27}$$

[2] We always assume that the solution to the optimal tax problem is interior and that second-order conditions are met.

The asterisks denote the compenated changes of the demand and supply functions.[3]

In order to interpret the first-order conditions, we will employ the following definitions.

Definition 1 *The* social *marginal value of transferring a marginal unit of income to individual* n *is:*

$$\lambda_n^* \equiv \Psi' \lambda_n + \eta t n \frac{\partial l_n}{\partial T} + (\eta\tau - \alpha\mu)\frac{\partial q_n}{\partial T}. \tag{28}$$

Definition 2 *The marginal cost of public funds is:*

$$MCF \equiv \eta/\bar{\lambda}^*, \quad \bar{\lambda}^* \equiv \int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n). \tag{29}$$

Definition 3 *The distributional characteristics of labor income* ξ_l*, polluting goods consumption* ξ_q*, and environmental quality* ξ_E *are defined as:*

$$\xi_l \equiv -\frac{\int_{\mathcal{N}} \lambda_n^* z_n \mathrm{d}F(n) - \int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} z_n \mathrm{d}F(n)}{\int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} z_n \mathrm{d}F(n)} = -\frac{\mathrm{cov}\left[\lambda_n^*, z_n\right]}{\bar{\lambda}^* \bar{z}} > 0, \tag{30}$$

$$\xi_q \equiv -\frac{\int_{\mathcal{N}} \lambda_n^* q_n \mathrm{d}F(n) - \int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} q_n \mathrm{d}F(n)}{\int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} q_n \mathrm{d}F(n)} = -\frac{\mathrm{cov}\left[\lambda_n^*, q_n\right]}{\bar{\lambda}^* \bar{q}}, \tag{31}$$

$$\xi_E \equiv -\frac{\int_{\mathcal{N}} \lambda_n^* \frac{u_E}{u_c} \mathrm{d}F(n) - \int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n)}{\int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n)} = -\frac{\mathrm{cov}\left[\lambda_n^*, \frac{u_E}{u_c}\right]}{\bar{\lambda}^* \overline{\frac{u_E}{u_c}}}, \tag{32}$$

where $\bar{z} \equiv \int_{\mathcal{N}} z_n \mathrm{d}F(n)$, $\bar{q} \equiv \int_{\mathcal{N}} q_n \mathrm{d}F(n)$, $\overline{\frac{u_E}{u_c}} \equiv \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n)$.

Definition 4 *The compensated elasticities of labor supply and polluting commodity demand with respect to the income tax, the corrective tax and environmental quality are defined as:*

$$\varepsilon_{lt} \equiv \frac{\partial l_n^*}{\partial t} \frac{1-t}{l_n} < 0, \tag{33}$$

$$\varepsilon_{qt} \equiv \frac{\partial q_n^*}{\partial t} \frac{1-t}{q_n}, \tag{34}$$

$$\varepsilon_{l\tau} \equiv \frac{\partial l_n^*}{\partial \tau} \frac{1+\tau}{l_n}, \tag{35}$$

$$\varepsilon_{q\tau} \equiv \frac{\partial q_n^*}{\partial \tau} \frac{1+\tau}{q_n} < 0, \tag{36}$$

$$\varepsilon_{lE} \equiv \frac{\partial l_n^*}{\partial E} \frac{E}{l_n}, \tag{37}$$

$$\varepsilon_{qE} \equiv \frac{\partial q_n^*}{\partial E} \frac{E}{q_n}. \tag{38}$$

Proposition 1 *(First-best optimum) In first best, all redistribution occurs through individualized lump-sum taxes, the marginal income tax rate is set to zero* ($t = 0$)*, the marginal cost of public funds equals unity* ($MCF = 1$)*, and the optimal corrective tax satisfies the first-best Pigouvian tax rate:*

$$\tau = \frac{\alpha\mu}{\bar{\lambda}^*}. \tag{39}$$

Moreover, the Pigouvian tax sustains a first-best level of environmental quality:

$$N \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n) = \frac{\mu}{\eta}. \tag{40}$$

Proof. When individualized lump-sum transfers and taxes are available, all inequality can be eliminated so as to equalize the social marginal value of income (λ_n^*) across agents. Consequently all distributional terms are zero ($\xi_l = \xi_q = \xi_E = 0$). Moreover, from (18) follows $MCF = 1$. Substitution in (54), (62) and (68) yields:

$$0 = \frac{t}{1-t}\left(-\overline{\varepsilon_{lt}}\right) + \frac{(\tau - \alpha\mu/\bar{\lambda}^*)}{1+\tau}\left(-\overline{\gamma\varepsilon_{qt}}\right). \tag{41}$$

[3] To compute the income effect of the change in environmental quality, the property has been used that $\frac{u_E}{u_c}$ measures the marginal change in (virtual) income when environmental quality improves by one unit, see **?**

$$0 = \frac{t}{1-t}\left(-\frac{\overline{\varepsilon_{l\tau}}}{\bar{\gamma}}\right) + \frac{\tau - \alpha\mu/\bar{\lambda}^*}{1+\tau}\left(-\frac{\overline{\gamma\varepsilon_{q\tau}}}{\bar{\gamma}}\right), \tag{42}$$

$$N\int_{\mathcal{N}} \frac{u_E}{u_c}\mathrm{d}F(n) = \frac{\mu}{\eta} - \delta N\left[\frac{t}{1-t}\overline{\varepsilon_{lE}} + \left(\frac{\tau - \alpha\mu/\bar{\lambda}^*}{1+\tau}\right)\overline{\gamma\varepsilon_{qE}}\right]. \tag{43}$$

Solving the first two equations yields $t = 0$ and $\tau = \alpha\mu/\bar{\lambda}^*$. From the last equation follows that $N\int_{\mathcal{N}} \frac{u_E}{u_c}\mathrm{d}F(n) = \frac{\mu}{\eta}$. ■

Proposition 2 *(Second-best full optimum) The policy rules for the optimal transfer, income tax, pollution tax and environmental quality are given by:*

$$MCF = 1, \tag{44}$$

$$\xi_l = \frac{t}{1-t}\left(-\overline{\varepsilon_{lt}}\right) + \frac{(\tau - \alpha\mu/\bar{\lambda}^*)}{1+\tau}\left(-\overline{\gamma\varepsilon_{qt}}\right), \tag{45}$$

$$\xi_q = \frac{t}{1-t}\left(-\frac{\overline{\varepsilon_{l\tau}}}{\bar{\gamma}}\right) + \frac{(\tau - \alpha\mu/\bar{\lambda}^*)}{1+\tau}\left(-\frac{\overline{\gamma\varepsilon_{q\tau}}}{\bar{\gamma}}\right). \tag{46}$$

$$(1-\xi_E)N\int_{\mathcal{N}} \frac{u_E}{u_c}\mathrm{d}F(n) = \frac{\mu}{\eta} + \delta N\left(\frac{t}{1-t}(-\overline{\varepsilon_{lE}}) + \frac{(\tau - \alpha\mu/\bar{\lambda}^*)}{1+\tau}(-\overline{\gamma\varepsilon_{qE}})\right), \tag{47}$$

where $\gamma_n \equiv \frac{(1+\tau)q_n}{(1-t)nl_n}$ *is the net expenditure share of polluting commodities in net labor income,* $\bar{\gamma} \equiv \left[\int_{\mathcal{N}} \gamma_n nl_n \mathrm{d}F(n)\right] \times \left[\int_{\mathcal{N}} nl_n \mathrm{d}F(n)\right]^{-1}$ *denotes the income-weighted average of* γ_n, $\delta \equiv (1-t)\int_{\mathcal{N}} nl_n \mathrm{d}F(n)/E$ *measures the ratio of net labor income to environmental quality, and* $\overline{\varepsilon_{xj}} \equiv \left[\int_{\mathcal{N}} \varepsilon_{xj} nl_n \mathrm{d}F(n)\right]\left[\int_{\mathcal{N}} nl_n \mathrm{d}F(n)\right]^{-1}$ *is the income-weighted average of the elasticity* ε_{xj}, $x = l, q$, $j = t, \tau, E$.

Proof. Optimal transfers – Substituting definition 2 for MCF in the first-order condition of the lump-sum transfer in equation (18) yields

$$\int_{\mathcal{N}}\left[\underbrace{\Psi'\lambda_n + \eta tn\frac{\partial l_n}{\partial T} + (\eta\tau - \alpha\mu)\frac{\partial q_n}{\partial T}}_{=\lambda_n^*}\right]\mathrm{d}F(n) = \eta. \tag{48}$$

Rewriting gives (44).

Optimal income taxes – Substituting the Slutsky equations (22) and (23) in equation (19) yields:

$$\frac{\partial\mathcal{L}}{\partial t} = \int_{\mathcal{N}}\left[-nl_n\Psi'\lambda_n + \eta nl_n + \eta tn\left(\frac{\partial l_n^*}{\partial t} - nl_n\frac{\partial l_n}{\partial T}\right) + (\eta\tau - \alpha\mu)\left(\frac{\partial q_n^*}{\partial t} - nl_n\frac{\partial q_n}{\partial T}\right)\right]\mathrm{d}F(n) = 0. \tag{49}$$

Rewriting gives:

$$\frac{\partial\mathcal{L}}{\partial t} = \int_{\mathcal{N}}\left[-\underbrace{\left(\Psi'\lambda_n + \eta tn\frac{\partial l_n}{\partial T} + (\eta\tau - \alpha\mu)\frac{\partial q_n}{\partial T}\right)}_{=\lambda_n^*}nl_n + \eta nl_n + \eta tn\frac{\partial l_n^*}{\partial t} + (\eta\tau - \alpha\mu)\frac{\partial q_n^*}{\partial t}\right]\mathrm{d}F(n) = 0. \tag{50}$$

Next, multiplying the third term with $1 = \frac{(1-t)}{(1-t)}\frac{l_n}{l_n}$ and the fourth term with $1 = \frac{(1+\tau)}{(1+\tau)}\frac{q_n}{q_n}\frac{(1-t)}{(1-t)}\frac{nl_n}{nl_n}$ and rewriting yields:

$$\frac{\partial\mathcal{L}}{\partial t} = \int_{\mathcal{N}}\left[-\lambda_n^* nl_n + \eta nl_n + \eta\frac{t}{1-t}nl_n\frac{\partial l_n^*}{\partial t}\frac{1-t}{l_n} + \eta\frac{\left(\tau - \frac{\alpha\mu}{\eta}\right)}{1+\tau}nl_n\frac{(1+\tau)q_n}{(1-t)nl_n}\frac{\partial q_n^*}{\partial t}\frac{1-t}{q_n}\right]\mathrm{d}F(n) = 0. \tag{51}$$

Substitute the definitions for the compensated elasticities (33) and (34) and the definition of the share $\gamma_n \equiv \frac{(1+\tau)q_n}{(1-t)nl_n}$ to find:

$$\frac{\partial\mathcal{L}}{\partial t} = \int_{\mathcal{N}}\left[-\lambda_n^* nl_n + \eta nl_n + \eta\frac{t}{1-t}nl_n\varepsilon_{lt} + \eta\frac{\left(\tau - \frac{\alpha\mu}{\eta}\right)}{1+\tau}nl_n\gamma_n\varepsilon_{qt}\right]\mathrm{d}F(n) = 0. \tag{52}$$

Dividing the equation by $\eta\int_{\mathcal{N}} nl_n\mathrm{d}F(n)$ gives:

$$1 - \frac{\int_{\mathcal{N}}\lambda_n^*\mathrm{d}F(n)}{\eta}\frac{\int_{\mathcal{N}}\lambda_n^* nl_n\mathrm{d}F(n)}{\int_{\mathcal{N}}\lambda_n^*\mathrm{d}F(n)\int_{\mathcal{N}} nl_n\mathrm{d}F(n)} + \frac{t}{1-t}\frac{\int_{\mathcal{N}} nl_n\varepsilon_{lt}\mathrm{d}F(n)}{\int_{\mathcal{N}} nl_n\mathrm{d}F(n)} + \frac{\left(\tau - \frac{\alpha\mu}{\eta}\right)}{1+\tau}\frac{\int_{\mathcal{N}} nl_n\gamma_n\varepsilon_{qt}\mathrm{d}F(n)}{\int_{\mathcal{N}} nl_n\mathrm{d}F(n)} = 0. \tag{53}$$

Using the definition for the marginal cost of funds (29) and distributional characteristic (30) gives:

$$1-\frac{1}{MCF}+\frac{\xi_l}{MCF}=\frac{t}{1-t}(-\overline{\varepsilon_{lt}})+\frac{\left(\tau-\frac{\alpha\mu/\bar{\lambda}^*}{MCF}\right)}{1+\tau}(-\overline{\gamma\varepsilon_{qt}}). \tag{54}$$

Substituting (44) gives (45).

Optimal pollution taxes – Substituting the Slutsky equations (24) and (25) in equation (20) yields:

$$\frac{\partial\mathcal{L}}{\partial\tau}=\int_{\mathcal{N}}\left[-q_n\Psi'\lambda_n+\eta q_n+\eta tn\left(\frac{\partial l_n^*}{\partial\tau}-q_n\frac{\partial l_n}{\partial T}\right)+(\eta\tau-\alpha\mu)\left(\frac{\partial q_n^*}{\partial\tau}-q_n\frac{\partial q_n}{\partial T}\right)\right]\mathrm{d}F(n)=0, \tag{55}$$

Rewriting gives:

$$\frac{\partial\mathcal{L}}{\partial\tau}=\int_{\mathcal{N}}\left[-\underbrace{\left(\Psi'\lambda_n+\eta tn\frac{\partial l_n}{\partial T}+(\eta\tau-\alpha\mu)\frac{\partial q_n}{\partial T}\right)}_{=\lambda_n^*}q_n+\eta q_n+\eta tn\frac{\partial l_n^*}{\partial\tau}+(\eta\tau-\alpha\mu)\frac{\partial q_n^*}{\partial\tau}\right]\mathrm{d}F(n)=0. \tag{56}$$

Next, multiplying the third term with $1=\frac{(1+\tau)}{(1+\tau)}\frac{q_n}{q_n}\frac{(1-t)}{(1-t)}\frac{l_n}{l_n}$ and the fourth term with $1=\frac{(1+\tau)}{(1+\tau)}\frac{q_n}{q_n}$ and rewriting yields:

$$\frac{\partial\mathcal{L}}{\partial\tau}=\int_{\mathcal{N}}\left[-\lambda_n^*q_n+\eta q_n+\eta\frac{t}{1-t}q_n\frac{(1-t)nl_n}{(1+\tau)q_n}\frac{\partial l_n^*}{\partial\tau}\frac{1+\tau}{l_n}+\eta\frac{\left(\tau-\frac{\alpha\mu}{\eta}\right)}{1+\tau}q_n\frac{\partial q_n^*}{\partial\tau}\frac{1+\tau}{q_n}\right]\mathrm{d}F(n)=0. \tag{57}$$

Substituting the definitions for the compensated elasticities (35) and (36) and the definition of the share $\gamma_n\equiv\frac{(1+\tau)q_n}{(1-t)nl_n}$ gives:

$$\frac{\partial\mathcal{L}}{\partial\tau}=\int_{\mathcal{N}}\left[-\lambda_n^*q_n+\eta q_n+\eta\frac{t}{1-t}q_n\frac{\varepsilon_{l\tau}}{\gamma_n}+\eta\frac{\left(\tau-\frac{\alpha\mu}{\eta}\right)}{1+\tau}q_n\varepsilon_{q\tau}\right]\mathrm{d}F(n)=0. \tag{58}$$

Dividing the equation by $\eta\int_{\mathcal{N}}nl_n\mathrm{d}F(n)$ gives:

$$\left(1-\frac{\int_{\mathcal{N}}\lambda_n^*\mathrm{d}F(n)}{\eta}\frac{\int_{\mathcal{N}}\lambda_n^*q_n\mathrm{d}F(n)}{\int_{\mathcal{N}}\lambda_n^*\mathrm{d}F(n)\int_{\mathcal{N}}q_n\mathrm{d}F(n)}\right)\frac{(1-t)}{(1+\tau)}\frac{\int_{\mathcal{N}}\gamma_nnl_n\mathrm{d}F(n)}{\int_{\mathcal{N}}nl_n\mathrm{d}F(n)}+\frac{(1-t)}{(1+\tau)}\frac{t}{1-t}\frac{\int_{\mathcal{N}}nl_n\varepsilon_{l\tau}\mathrm{d}F(n)}{\int_{\mathcal{N}}nl_n\mathrm{d}F(n)} \tag{59}$$

$$+\quad\frac{\left(\tau-\frac{\alpha\mu}{\eta}\right)}{1+\tau}\frac{(1-t)}{(1+\tau)}\frac{\int_{\mathcal{N}}\gamma_nnl_n\varepsilon_{q\tau}\mathrm{d}F(n)}{\int_{\mathcal{N}}nl_n\mathrm{d}F(n)}=0.$$

$$\left(1-\frac{1-\xi_q}{MCF}\right)\frac{(1-t)}{(1+\tau)}\bar{\gamma}+\frac{(1-t)}{(1+\tau)}\frac{t}{1-t}\overline{\varepsilon_{l\tau}}+\frac{\left(\tau-\frac{\alpha\mu}{\eta}\right)}{1+\tau}\frac{(1-t)}{(1+\tau)}\frac{\int_{\mathcal{N}}\gamma_nnl_n\varepsilon_{q\tau}\mathrm{d}F(n)}{\int_{\mathcal{N}}nl_n\mathrm{d}F(n)}=0. \tag{60}$$

Using $q_n=\frac{(1-t)}{(1+\tau)}\gamma_nnl_n$, the definition for the marginal cost of funds (29), the distributional characteristic (31), and dividing by $\frac{(1-t)}{(1+\tau)}$ gives:

$$\left(1-\frac{1-\xi_q}{MCF}\right)\bar{\gamma}+\frac{t}{1-t}\frac{\int_{\mathcal{N}}\varepsilon_{l\tau}nl_n\mathrm{d}F(n)}{\int_{\mathcal{N}}nl_n\mathrm{d}F(n)}+\frac{\left(\tau-\frac{\alpha\mu}{\eta}\right)}{1+\tau}\frac{\int_{\mathcal{N}}\gamma_n\varepsilon_{q\tau}nl_n\mathrm{d}F(n)}{\int_{\mathcal{N}}nl_n\mathrm{d}F(n)}=0. \tag{61}$$

Rewriting yields:

$$1-\frac{1}{MCF}+\frac{\xi_q}{MCF}=\frac{t}{1-t}\left(-\frac{\overline{\varepsilon_{l\tau}}}{\bar{\gamma}}\right)+\frac{\left(\tau-\frac{\alpha\mu/\bar{\lambda}^*}{MCF}\right)}{1+\tau}\left(-\frac{\overline{\gamma\varepsilon_{q\tau}}}{\bar{\gamma}}\right), \tag{62}$$

Substituting (44) gives (46).

Optimal environmental quality – Substituting the Slutsky equations (26) and (27) in equation (21) yields:

$$\frac{\partial\mathcal{L}}{\partial E}=\int_{\mathcal{N}}\left[\frac{u_E}{u_c}\Psi'\lambda_n-\frac{\mu}{N}+\eta tn\left(\frac{\partial l_n^*}{\partial E}+\frac{u_E}{u_c}\frac{\partial l_n}{\partial T}\right)+(\eta\tau-\alpha\mu)\left(\frac{\partial q_n^*}{\partial E}+\frac{u_E}{u_c}\frac{\partial q_n}{\partial T}\right)\right]\mathrm{d}F(n)=0. \tag{63}$$

Rewriting gives:

$$\frac{\partial\mathcal{L}}{\partial E}=\int_{\mathcal{N}}\left[\underbrace{\left(\Psi'\lambda_n+\eta tn\frac{\partial l_n}{\partial T}+(\eta\tau-\alpha\mu)\frac{\partial q_n}{\partial T}\right)}_{=\lambda_n^*}\frac{u_E}{u_c}-\frac{\mu}{N}+\eta tn\frac{\partial l_n^*}{\partial E}+(\eta\tau-\alpha\mu)\frac{\partial q_n^*}{\partial E}\right]\mathrm{d}F(n)=0. \tag{64}$$

Next, multiplying the third term with $1 = \frac{E}{E}\frac{(1-t)}{(1-t)}\frac{l_n}{l_n}$ and the fourth term with $1 = \frac{E}{E}\frac{(1+\tau)}{(1+\tau)}\frac{q_n}{q_n}\frac{(1-t)nl_n}{(1-t)nl_n}$ and rewriting yields:

$$\frac{\partial \mathcal{L}}{\partial E} = \int_{\mathcal{N}} \left[\lambda_n^* \frac{u_E}{u_c} - \frac{\mu}{N} + \eta \frac{t}{1-t} \frac{(1-t)nl_n}{E} \frac{\partial l_n^*}{\partial E} \frac{E}{l_n} + \eta \frac{\left(\tau - \frac{\alpha\mu}{\eta}\right)}{1+\tau} \frac{(1-t)nl_n}{E} \frac{(1+\tau)q_n}{(1-t)nl_n} \frac{\partial q_n^*}{\partial E} \frac{E}{q_n} \right] \mathrm{d}F(n) = 0. \quad (65)$$

Substitute the definitions for the compensated elasticities (37) and (38) and the definition of the share $\gamma_n \equiv \frac{(1+\tau)q_n}{(1-t)nl_n}$ to find:

$$\frac{\partial \mathcal{L}}{\partial E} = \int_{\mathcal{N}} \left[\lambda_n^* \frac{u_E}{u_c} - \frac{\mu}{N} + \eta \frac{t}{1-t} \frac{(1-t)}{E} \varepsilon_{lE} nl_n + \eta \frac{\left(\tau - \frac{\alpha\mu}{\eta}\right)}{1+\tau} \frac{(1-t)}{E} \gamma_n \varepsilon_{qE} nl_n \right] \mathrm{d}F(n) = 0. \quad (66)$$

Dividing the equation by $\eta \int_{\mathcal{N}} nl_n \mathrm{d}F(n)$ gives:

$$\frac{\int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n)}{\eta} \frac{\int_{\mathcal{N}} \lambda_n^* \frac{u_E}{u_c} \mathrm{d}F(n)}{\int_{\mathcal{N}} \lambda_n^* \mathrm{d}F(n) \int_{\mathcal{N}} nl_n \mathrm{d}F(n)} - \frac{\mu}{\eta \int_{\mathcal{N}} nl_n \mathrm{d}F(n) N} \quad (67)$$
$$+ \frac{t}{1-t} \frac{(1-t)}{E} \frac{\int_{\mathcal{N}} \varepsilon_{lE} nl_n \mathrm{d}F(n)}{\int_{\mathcal{N}} nl_n \mathrm{d}F(n)} + \frac{\left(\tau - \frac{\alpha\mu}{\eta}\right)}{1+\tau} \frac{(1-t)}{E} \frac{\int_{\mathcal{N}} \gamma_n \varepsilon_{qE} nl_n \mathrm{d}F(n)}{\int_{\mathcal{N}} nl_n \mathrm{d}F(n)} = 0$$

Using the definition for the marginal cost of funds (29), distributional characteristic (32) and $\delta \equiv (1-t) \int_{\mathcal{N}} nl_n \mathrm{d}F(n)/E$ gives:

$$\frac{(1-\xi_E)}{MCF} N \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n) = \frac{\mu}{\eta} - \delta N \left[\frac{t}{1-t} \overline{\varepsilon_{lE}} + \left(\frac{\tau - \frac{\alpha\mu/\bar{\lambda}^*}{MCF}}{1+\tau} \right) \overline{\gamma\varepsilon_{qE}} \right], \quad (68)$$

Substituting (44) gives (47). ■

Proposition 3 *(Second-best constrained optimum) When the government cannot optimize non-individualized lump-sum transfers, the policy rules for the optimal income tax, pollution tax and environmental quality are given by:*

$$1 - \frac{1}{MCF} + \frac{\xi_l}{MCF} = \frac{t}{1-t}(-\overline{\varepsilon_{lt}}) + \frac{\left(\tau - \frac{\alpha\mu/\bar{\lambda}^*}{MCF}\right)}{1+\tau}(-\overline{\gamma\varepsilon_{qt}}), \quad (69)$$

$$1 - \frac{1}{MCF} + \frac{\xi_q}{MCF} = \frac{t}{1-t}\left(-\frac{\overline{\varepsilon_{l\tau}}}{\bar{\gamma}}\right) + \frac{\left(\tau - \frac{\alpha\mu/\bar{\lambda}^*}{MCF}\right)}{1+\tau}\left(-\frac{\overline{\gamma\varepsilon_{q\tau}}}{\bar{\gamma}}\right), \quad (70)$$

$$(1-\xi_E) N \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n) = MCF \cdot \left[\frac{\mu}{\eta} + \delta N \left(\frac{t}{1-t}(-\overline{\varepsilon_{lE}}) + \frac{\left(\tau - \frac{\alpha\mu/\bar{\lambda}^*}{MCF}\right)}{1+\tau}(-\overline{\gamma\varepsilon_{qE}}) \right) \right]. \quad (71)$$

Proof. See equations (54), (62) and (68) in the proof of previous Proposition. These correspond to the optimal tax expressions for a given level of the non-individualized lump-sum tax T. ■

Corollary 1 *If preferences are given by*

$$u_n \equiv \upsilon(c_n, q_n) - h(l_n) + \Gamma(E), \quad \upsilon_c, \upsilon_q, h', \Gamma' > 0, \quad \upsilon_{cc}, \upsilon_{qq}, -h'', \Gamma'' < 0, \quad \forall n, \quad (72)$$

where $\upsilon(\cdot)$ denotes total real consumption from clean and dirty commodities, and $\upsilon(.)$ is a linear homogeneous sub-utility function over clean and dirty commodities, then the optimal income tax is given by $\frac{t}{1-t} = \frac{\xi_l}{-\bar{\varepsilon}_{lt}}$, the modified Pigouvian tax equals the first-best Pigouvian tax, $\tau = \alpha\mu/\bar{\lambda}^*$, and environmental quality follows the first-best Samuelson rule, $N \int_{\mathcal{N}} \frac{u_E}{u_c} \mathrm{d}F(n) = \frac{\mu}{\eta}$.

Proof. Marginal utility of income is constant, due to the linear homogeneity of $\upsilon(\cdot)$. Due to linear homogeneity, the first-order condition $\frac{u_q}{u_c} = 1 + \tau$ can be written as $q_n = \phi(1+\tau)c_n$ where $\phi(\cdot)$ is some function with $\phi'(\cdot) < 0$. Use this in the definition for λ_n gives $\lambda_n = u_c(c_n, \phi(1+\tau)c_n)$. If the utility function u is linear homogeneous in c_n and q_n the derivatives of u are homogeneous of degree zero. Hence, $u_{cc}c_n + u_{cq}q_n = 0$ and $u_{cc} + u_{cq}\phi(1+\tau) = 0$. Taking the derivative $\frac{\partial \lambda_n}{\partial c_n} = u_{cc} + u_{cq}\phi(1+\tau) = 0$ demonstrates that income effects are absent. Therefore, $\frac{u_E}{u_c}$ is constant, so that $\xi_E = 0$, cf. (32). Furthermore, $\varepsilon_{lt} = \gamma\varepsilon_{l\tau}$, which follows from totally differentiating the first-order conditions (8), (9), and the utility function (1), and setting the change in utility to zero in order to find the compensated elasticities. To prove this, we write the household problem as a two-stage budgeting problem. In the first stage, individuals choose aggregate consumption υ_n and labor supply l_n to maximize utility u_n subject to budget constraint $p(\tau)\upsilon_n = (1-t)nl_n + T$, where $p(\tau)$ is the real price index for aggregate private consumption υ_n. This gives first-order condition $\frac{-u_l}{u_\upsilon} = \frac{(1-t)n}{p(\tau)}$. To find the elasticities, linearize the first-order condition $\left(\frac{u_{ll}l_n}{u_l} - \frac{u_{\upsilon l}l_n}{u_\upsilon}\right)\tilde{l}_n = \left(\frac{u_{\upsilon\upsilon}\upsilon_n}{u_\upsilon} - \frac{u_{l\upsilon}\upsilon_n}{u_l}\right)\tilde{\upsilon}_n - \tilde{t} - \tilde{p}$, where a tilde ($\tilde{}$) denotes a relative change, e.g., $\tilde{l}_n \equiv \mathrm{d}l_n/l_n$, except for the tax rates, where $\tilde{t} \equiv \mathrm{d}t/(1-t)$ and $\tilde{\tau} \equiv \mathrm{d}\tau/(1+\tau)$. When utility does not change, we have the following linearized utility function $u_\upsilon \upsilon_n \tilde{\upsilon}_n + u_l l_n \tilde{l}_n = 0$. Using the first-order condition for labor supply yields $\theta_n \tilde{\upsilon}_n = \tilde{l}_n$, where $\theta_n \equiv \frac{(1-t)nl_n+T}{(1-t)nl_n}$. Solving for $\tilde{l}_n$ gives $\tilde{l}_n = -\varepsilon_n\left(\tilde{t}+\tilde{p}\right)$, and $\varepsilon_n \equiv \left(\frac{u_{ll}l_n}{u_l} - \frac{u_{\upsilon l}l_n}{u_\upsilon} - \frac{1}{\theta_n}\left(\frac{u_{\upsilon\upsilon}\upsilon_n}{u_\upsilon} - \frac{u_{l\upsilon}\upsilon_n}{u_l}\right)\right)^{-1}$. In the second stage, households choose to allocate their resources over clean and polluting commodities to maximize $\upsilon(c_n, q_n)$ subject to $p(\tau)\upsilon_n = c_n + (1+\tau)q_n$. This yields first-order condition $\frac{\upsilon_q}{\upsilon_c} = 1+\tau$. Since, sub-utility υ is homothetic, linearizing sub-utility υ_n gives $\tilde{\upsilon}_n = (1-\gamma)\tilde{c}_n + \gamma\tilde{q}_n$, where $\gamma \equiv \frac{(1+\tau)q_n}{(1-t)nl_n+T}$ is constant. The price index satisfies $p(\tau)\upsilon_n = c_n + (1+\tau)q_n$. It can be linearized to find $\tilde{p} + \tilde{\upsilon}_n = (1-\gamma)\tilde{c}_n + \gamma\tilde{q}_n + \gamma\tilde{\tau} = \gamma\tilde{\tau}$. Therefore, we find for the change in labor supply: $\tilde{l}_n = -\varepsilon_n\tilde{t} - \varepsilon_n\gamma\tilde{\tau}$, where ε_n is the compensated labor-supply elasticity. Consequently, we establish that $\varepsilon_{lt} \equiv -\varepsilon_n$ and $\varepsilon_{l\tau} \equiv -\varepsilon_n\gamma$. Finally, it can be derived that $\int_{\mathcal{N}}(1-\lambda_n^*)\left(\frac{1+\tau}{1-t}\right)q_n \mathrm{d}F(n) = \int_{\mathcal{N}}(1-\lambda_n^*)\gamma_n n l_n \mathrm{d}F(n)$, since $\gamma_n = \gamma \equiv \frac{(1+\tau)q_n}{(1-t)nl_n+T}$ is constant with homothetic preferences and using the first-order condition for T from (18). Substitution of all these results in the first-order conditions – equations (54), (62) and (68) – proves the proposition. ■

3 Optimal non-linear taxation

To solve for the optimal non-linear tax this Appendix proceeds as follows. First, we prove Lemma 1 of the paper. Second, we prove Proposition 3 of the paper.

3.1 Proof Lemma 1

Lemma 1 *The compensated elasticity of labor supply with respect to the marginal income tax rate is:*

$$\varepsilon_{lT'} \equiv \frac{\partial l_n^*}{\partial T'}\frac{1-T'(nl_n)}{l_n} = \frac{u_l/l_n}{u_{ll} + \left(\frac{u_l}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_l}{u_c}\right)u_{cl} + nu_l\frac{T''}{1-T'}} > 0. \tag{73}$$

The uncompensated elasticity of earnings with respect to the wage rate is:[4]

$$\varepsilon_{zn}^u \equiv \frac{\partial z_n}{\partial n}\frac{n}{z_n} = \frac{u_l/l_n + u_{ll} - \left(\frac{u_l}{u_c}\right)u_{cl}}{u_{ll} + \left(\frac{u_l}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_l}{u_c}\right)u_{cl} + nu_l\frac{T''}{1-T'}} > 0. \tag{74}$$

The compensated elasticity of polluting goods demand with respect to the marginal pollution tax rate is

$$\varepsilon_{q\tau'} \equiv -\frac{\partial q_n^*}{\partial \tau'}\frac{1-\tau'(q_n)}{q_n} = -\frac{u_q/q_n}{u_{qq} + \left(\frac{u_q}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_q}{u_c}\right)u_{cq} - \frac{\tau''}{1+\tau'}u_q} > 0. \tag{75}$$

The uncompensated (and uncompensated) cross elasticity of labor-supply with respect to demand for polluting goods, conditional on net income $\bar{y} \equiv \bar{z} - T(\bar{z})$, is

$$\left.\varepsilon_{ql}^u\right|_{\bar{y}} \equiv \left.\frac{\partial q_n}{\partial l_n}\frac{l_n}{q_n}\right|_{\bar{y}} = \frac{\left(\frac{u_q}{u_c}u_{cl} - u_{ql}\right)l_n/q_n}{u_{qq} + \left(\frac{u_q}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_q}{u_c}\right)u_{cq} - \frac{\tau''}{1+\tau'}u_q} \gtreqless 0. \tag{76}$$

3.1.1 Elasticities of labor and earnings supply

The household budget constraint under non-linear policy instruments is given by

$$c_n + q_n + \tau(q_n) = z_n - T(z_n), \quad \forall n. \tag{77}$$

Maximizing utility $u(c_n, q_n, z_n/n, E)$ subject to the household budget constraint yields the following necessary first-order conditions:

$$\frac{-u_l}{u_c} = (1-T'(z_n))\,n, \quad \frac{u_q}{u_c} = 1+\tau'(q_n), \quad \forall n. \tag{78}$$

[4] The uncompensated elasticity of earnings with respect to the wage rate is always positive given that earnings should be monotonic in skills at the optimal second-best allocation. The labor-supply elasticity with respect to the wage rate could be negative, however, due to off-setting income and substitution effects.

As in ?, we can define the following *shift function* L for labor supply while suppressing the indices n:

$$L\left(l,q,n,E,\theta,\rho\right) \equiv n\left(1-T'\left(nl\right)-\theta\right)u_c\left(nl-T\left(nl\right)-\theta\left(nl-nl_n\right)+\rho-q-\tau(q),q,l,E\right) \\ +u_l\left(nl-T\left(nl\right)-\theta\left(nl-nl_n\right)+\rho-q-\tau(q),l,q,E\right). \tag{79}$$

$L\left(l,q,n,E,\theta,\rho\right)$ measures a *shift* in the first-order condition for labor supply when one of the variables l,q,n,E,θ or ρ changes. θ is introduced to capture an exogenous increase in the marginal tax rate (i.e., for any level of earnings). ρ is introduced to retrieve the income effect when the household receives an exogenous amount of income ρ, irrespective of the amount of labor supplied. The first-order condition for labor supply of the household n is thus equivalent to $L\left(l,q,n,0,0\right)=0$. Introducing the term $\theta\left(nl-nl_n\right)$ has the following intuition. Suppose that we raise the marginal tax rate – irrespective of income level nl – and we evaluate the impact at nl_n (the optimum choice for l_n of household n), then this marginal tax increase does not change income, only the marginal incentives to supply labor. ρ represents the income effect: suppose that we give the household a marginal increase in income of ρ, starting from $\rho=0$, what will happen to labor supply?

We find the following partial derivatives, using the first-order condition $-u_l=n\left(1-T'\right)u_c$:

$$L_l\left(l,q,E,n,0,0\right)=u_{ll}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-2\frac{u_l}{u_c}u_{cl}+nu_l\frac{T''}{1-T'}, \tag{80}$$

$$L_q\left(l,q,E,n,0,0\right)=\frac{u_l}{u_c}\left(u_{cc}\frac{u_q}{u_c}-u_{cq}\right)+u_{lq}-u_{lc}\frac{u_q}{u_c}, \tag{81}$$

$$L_n\left(l,q,E,n,0,0\right)=\left(-\frac{u_l}{l}+nu_l\frac{T''}{1-T'}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-\left(\frac{u_l}{u_c}\right)u_{lc}\right)\frac{l}{n}, \tag{82}$$

$$L_E\left(l,q,E,n,0,0\right)=u_{lE}-\frac{u_l}{u_c}u_{cE}, \tag{83}$$

$$L_\theta\left(l,q,E,n,0,0\right)=-nu_c, \tag{84}$$

$$L_\rho\left(l,q,E,n,0,0\right)=n\left(1-T'\right)u_{cc}+u_{lc}=u_{lc}-\frac{u_l}{u_c}u_{cc}. \tag{85}$$

Now, by applying the implicit function theorem we derive that

$$\frac{\partial l}{\partial x}=-\frac{L_x}{L_l}, \quad x=q,n,\theta,\rho. \tag{86}$$

We thus obtain the uncompensated elasticity of labor supply w.r.t. commodity demand q:

$$\varepsilon_{lq}^u\equiv\frac{\partial l}{\partial q}\frac{q}{l}=-\frac{L_q}{L_l}\frac{q}{l}=\frac{q}{l}\frac{\frac{u_l}{u_c}\left(u_{cc}\frac{u_q}{u_c}-u_{cq}\right)+u_{lq}-u_{lc}\frac{u_q}{u_c}}{u_{ll}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-2\frac{u_l}{u_c}u_{cl}+nu_l\frac{T''}{1-T'}}. \tag{87}$$

The uncompensated *wage* elasticity of labor supply ε_{ln}^u is equal to:

$$\varepsilon_{ln}^u\equiv\frac{\partial l}{\partial n}\frac{n}{l}=-\frac{L_n}{L_l}\frac{n}{l}=\frac{u_l/l+\left(\frac{u_l}{u_c}\right)u_{lc}-\left(\frac{u_l}{u_c}\right)^2u_{cc}-nu_l\frac{T''}{1-T'}}{u_{ll}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-2\left(\frac{u_l}{u_c}\right)u_{cl}+nu_l\frac{T''}{1-T'}}. \tag{88}$$

Note further that the uncompensated *wage* elasticity of *earnings* supply ε_{zn} is equal to:

$$\varepsilon_{zn}^u\equiv\frac{\partial z}{\partial n}\frac{n}{z}=1+\varepsilon_{ln}=\frac{u_l/l+u_{ll}-\left(\frac{u_l}{u_c}\right)u_{cl}}{u_{ll}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-2\left(\frac{u_l}{u_c}\right)u_{cl}+nu_l\frac{T''}{1-T'}}. \tag{89}$$

The income elasticity of labor supply ε_l^I is defined as:

$$\varepsilon_l^I\equiv\left(1-T'\right)n\frac{\partial l}{\partial\rho}=-\left(1-T'\right)n\frac{L_\rho}{L_l}=\frac{\frac{-u_l}{u_c}\left(\frac{u_l}{u_c}u_{cc}-u_{lc}\right)}{u_{ll}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-2\left(\frac{u_l}{u_c}\right)u_{cl}+nu_l\frac{T''}{1-T'}}. \tag{90}$$

The compensated *wage* elasticity of labor supply ε_{ln} is defined residually by the Slutsky equation ($\varepsilon_{ln}\equiv\varepsilon_{ln}^u-\varepsilon_l^I$):

$$\varepsilon_{ln}\equiv\frac{\partial l^*}{\partial n}\frac{n}{l}=\frac{u_l/l-nu_l\frac{T''}{1-T'}}{u_{ll}+\left(\frac{u_l}{u_c}\right)^2u_{cc}-2\left(\frac{u_l}{u_c}\right)u_{cl}-nu_l\frac{T''}{1-T'}}. \tag{91}$$

Next, the compensated *tax* elasticity of labor supply $\varepsilon_{lT'}$ is:

$$\varepsilon_{lT'} \equiv -\frac{\partial l^*}{\partial \theta}\frac{(1-T')}{l} = \frac{L_\theta}{L_l}\frac{(1-T')}{l} = \frac{u_l/l}{u_{ll} + \left(\frac{u_l}{u_c}\right)^2 u_{cc} - 2\frac{u_l}{u_c}u_{cl} + nu_l\frac{T''}{1-T'}}. \tag{92}$$

The compensated *tax* elasticity of earnings supply $\varepsilon_{zT'}$ equals the compensated tax elasticity of labor supply (since the wage rate n is not affected by an increase in marginal taxes, only labor supply):

$$\varepsilon_{zT'} \equiv -\frac{\partial z^*}{\partial \theta}\frac{(1-T')}{z} = \varepsilon_{lT'} = \frac{u_l/l}{u_{ll} + \left(\frac{u_l}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_l}{u_c}\right)u_{cl} + nu_l\frac{T''}{1-T'}}. \tag{93}$$

Finally, we obtain the uncompensated labor-supply elasticity with respect to environmental quality E:

$$\varepsilon_{lE} \equiv \frac{\partial l}{\partial E}\frac{E}{l} = -\frac{L_E}{L_l}\frac{E}{l} = \frac{\left(\frac{u_l}{u_c}u_{cE} - u_{lE}\right)E/l}{u_{ll} + \left(\frac{u_l}{u_c}\right)^2 u_{cc} - 2\frac{u_l}{u_c}u_{cl} + nu_l\frac{T''}{1-T'}}. \tag{94}$$

3.1.2 Elasticities of commodity demands

We add price p for polluting good q to retrieve the income effect in demand for polluting goods in a clear and transparant way. Later we will normalize the price p back to one. The household budget constraint is then given by:

$$c_n + pq_n + \tau(pq_n) = z_n - T(z_n), \quad \forall n. \tag{95}$$

Maximizing utility $u(c_n, q_n, z_n/n, E)$ subject to the household budget constraint yields the following necessary first-order conditions:

$$\frac{-u_l}{u_c} = (1 - T'(z_n))\, n, \quad \frac{u_q}{u_c} = p(1 + \tau'(pq_n)), \quad \forall n. \tag{96}$$

Follow the same procedure as above. Hence, we can define the following *shift function* Q for the demand of the poluting commodity while suppressing the indices n:

$$\begin{aligned} Q\left(l, q, n, E, p, \theta, \rho\right) \equiv\, & p\left(1 + \tau'\left(pq\right) + \theta\right) u_c\left(nl - T\left(nl\right) - pq - \tau(pq) - \theta\left(pq - pq_n\right) + \rho, l, q, E\right) \\ & - u_q\left(nl - T\left(nl\right) - pq - \tau(pq) - \theta\left(pq - pq_n\right) + \rho, l, q, E\right). \end{aligned} \tag{97}$$

$Q\left(l, q, n, E, p, \theta, \rho\right)$ measures a *shift* in the first-order condition for polluting goods demand when one of the variables l, q, n, p, θ or ρ changes, while labor supply remains . θ and ρ have the same role as in the shift-function for labor. θ is introduced to capture an exogenous increase in the marginal tax rate on the polluting commodity (i.e. for any level of commodity demand). ρ is introduced to retrieve the income effect when the household receives an exogenous amount of income ρ, irrespective of polluting goods demand. The first-order condition for pollution goods demand q of the household n is thus equivalent to $Q\left(l, q, n, E, p, 0, 0\right) = 0$.

We find the following partial derivatives, using the first-order condition $u_q = u_c p(1 + \tau'(pq_n))$:

$$Q_l\left(l, q, n, E, p, 0, 0\right) = \frac{-u_l}{u_c}\left(\frac{u_q}{u_c}u_{cc} - u_{qc}\right) + \frac{u_q}{u_c}u_{cl} - u_{ql}, \tag{98}$$

$$Q_q\left(l, q, n, E, p, 0, 0\right) = -u_{qq} - \left(\frac{u_q}{u_c}\right)^2 u_{cc} + 2\left(\frac{u_q}{u_c}\right)u_{cq} + \frac{\tau''}{1+\tau'}pu_q, \tag{99}$$

$$Q_n\left(l, q, n, E, p, 0, 0\right) = -\left(\frac{u_q}{u_c}u_{cc} - u_{qc}\right)\frac{u_l}{u_c}\frac{l}{n}, \tag{100}$$

$$Q_E\left(l, q, n, E, p, 0, 0\right) = \frac{u_q}{u_c}u_{cE} - u_{qE}, \tag{101}$$

$$Q_p\left(l, q, n, E, p, 0, 0\right) = \left(\frac{u_q}{q} - \left(\frac{u_q}{u_p}\right)^2 u_{cc} + u_{qc}\frac{u_q}{u_c} + \frac{\tau''}{1+\tau'}pu_q\right)\frac{q}{p}, \tag{102}$$

$$Q_\theta\left(l, q, n, E, p, 0, 0\right) = pu_c, \tag{103}$$

$$Q_\rho\left(l, q, n, E, p, 0, 0\right) = p(1+\tau')u_{cc} - u_{qc} = \frac{u_q u_{cc}}{u_c} - u_{qc}. \tag{104}$$

Now, by applying the implicit function theorem we find:

$$\frac{\partial q}{\partial x} = -\frac{Q_x}{Q_q}, \quad x = l, n, E, p, \theta, \rho. \tag{105}$$

Hence, from this we can calculate the elasticities. From here on, we assume again that $p=1$ everywhere.

The uncompensated elasticity of commodity demands with respect to labor supply is given by:

$$\varepsilon^u_{ql} \equiv \frac{\partial q}{\partial l}\frac{l}{q} = -\frac{Q_l}{Q_q}\frac{l}{q} = \frac{l}{q}\frac{\frac{-u_l}{u_c}\left(\frac{u_q}{u_c}u_{cc}-u_{qc}\right)+\frac{u_q}{u_c}u_{cl}-u_{ql}}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}u_q}. \tag{106}$$

The uncompensated elasticity of commodity demands with respect to the wage rate is:

$$\varepsilon^u_{qn} \equiv \frac{\partial q}{\partial n}\frac{n}{q} = -\frac{Q_n}{Q_q}\frac{n}{q} = \frac{l}{q}\frac{\frac{-u_l}{u_c}\left(\frac{u_q}{u_c}u_{cc}-u_{qc}\right)}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}u_q}. \tag{107}$$

The uncompensated price elasticity of commodity demand ε_{qp} is equal to:

$$\varepsilon^u_{qp} \equiv -\frac{\partial q}{\partial p}\frac{p}{q} = \frac{p}{q}\frac{Q_p}{Q_q} = -\frac{\frac{u_q}{q}-\left(\frac{u_q}{u_p}\right)^2 u_{cc}+u_{qc}\frac{u_q}{u_c}+\frac{\tau''}{1+\tau'}u_q}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}u_q}. \tag{108}$$

The income elasticity of commodity demand ε^I_q is defined as:

$$\varepsilon^I_q \equiv (1+\tau')\,p\frac{\partial q}{\partial \rho} = -(1+\tau')\,p\frac{Q_\rho}{Q_q} = \frac{\frac{u_q}{u_c}\left(\frac{u_q}{u_c}u_{cc}-u_{qc}\right)}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}u_q}. \tag{109}$$

Hence, using the Slutsky equation ($\varepsilon^u_{qp}=\varepsilon_{qp}-\varepsilon^I_q$) the compe=nsated price elasticity of demand for commodity q is determined residually:

$$\varepsilon_{qp} = -\frac{\partial q^*}{\partial p}\frac{p}{q} = -\frac{u_q/q+\frac{\tau''}{1+\tau'}pu_q}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}u_q}. \tag{110}$$

The compensated tax elasticity of commodity demand $\varepsilon_{q\tau'}$ is equal to:

$$\varepsilon_{q\tau'} \equiv -\frac{\partial q^*}{\partial \theta}\frac{(1+\tau')}{q} = \frac{(1+\tau')}{q}\frac{Q_\theta}{Q_q} = -\frac{u_q/q}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}u_q}. \tag{111}$$

And, the uncompensated elasticity of demand of polluting goods with respect to environmental quality is:

$$\varepsilon^u_{qE} \equiv \frac{\partial q}{\partial E}\frac{E}{q} = -\frac{Q_E}{Q_q}\frac{E}{q} = \frac{\left(\frac{u_q}{u_c}u_{cE}-u_{qE}\right)E/q}{u_{qq}+\left(\frac{u_q}{u_c}\right)^2 u_{cc}-2\left(\frac{u_q}{u_c}\right)u_{cq}-\frac{\tau''}{1+\tau'}pu_q}. \tag{112}$$

3.1.3 Elasticities of conditional commodity demands

Finally, we evaluate the first-order condition for q and c for a *given* level of net income $\bar{y}\equiv\bar{z}-T(\bar{z})$. First, the shift function for commodity demands is now modified to:

$$Q\left(l,q,p,n,0,0|\bar{y}\right) \equiv p\left(1+\tau'\left(pq\right)\right)u_c\left(\bar{z}-T\left(\bar{z}\right)-\left(pq+\tau(pq)\right),l,q,E\right)-u_q\left(\bar{z}-T\left(\bar{z}\right)-\left(pq+\tau(pq)\right),l,q,E\right). \tag{113}$$

Therefore, we can derive:

$$Q_q\left(l,q,n,E,p,0,0|\bar{y}\right) = -u_{qq}-\left(\frac{u_q}{u_c}\right)^2 u_{cc}+2\left(\frac{u_q}{u_c}\right)u_{cq}+\frac{\tau''}{1+\tau'}pu_q, \tag{114}$$

$$Q_l\left(l,q,n,E,p,0,0|\bar{y}\right) = p(1+\tau')u_{cl}-u_{ql} = \frac{u_q}{u_c}u_{cl}-u_{ql}, \tag{115}$$

$$Q_n\left(l,q,n,E,p,0,0|\bar{y}\right) = -p(1+\tau')u_{cl}\frac{l}{n}+u_{ql}\frac{l}{n} = -\left(\frac{u_q}{u_c}u_{cl}-u_{ql}\right)\frac{l}{n} \tag{116}$$

$$Q_E\left(l,q,n,E,p,0,0\right) = \frac{u_q}{u_c}u_{cE}-u_{qE}, \tag{117}$$

where we obtained the derivative Q_n by substituting $l = z/n$ in the modified shift function. Hence, by applying the envelope theorem we find (assuming $p = 1$ throughout):

$$\varepsilon^u_{ql}\big|_{\bar{y}} \equiv \frac{\partial q}{\partial l}\frac{l}{q}\bigg|_{\bar{y}} = -\frac{Q_l}{Q_q}\frac{l}{q}\bigg|_{\bar{y}} = \frac{\left(\frac{u_q}{u_c}u_{cl} - u_{ql}\right) l/q}{u_{qq} + \left(\frac{u_q}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_q}{u_c}\right) u_{cq} - \frac{\tau''}{1+\tau'}u_q}, \tag{118}$$

$$\varepsilon^u_{qn}\big|_{\bar{y}} \equiv \frac{\partial q}{\partial n}\frac{n}{q}\bigg|_{\bar{y}} = -\frac{Q_n}{Q_q}\frac{n}{q}\bigg|_{\bar{y}} = -\frac{\left(\frac{u_q}{u_c}u_{cl} - u_{ql}\right) l/q}{u_{qq} + \left(\frac{u_q}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_q}{u_c}\right) u_{cq} - \frac{\tau''}{1+\tau'}u_q} = -\,\varepsilon_{ql}\big|_{\bar{y}}, \tag{119}$$

$$\varepsilon^u_{qE}\big|_{\bar{y}} \equiv \frac{\partial q}{\partial E}\frac{E}{q}\bigg|_{\bar{y}} = -\frac{Q_E}{Q_q}\frac{E}{q}\bigg|_{\bar{y}} = \frac{\left(\frac{u_q}{u_c}u_{cE} - u_{qE}\right) E/q}{u_{qq} + \left(\frac{u_q}{u_c}\right)^2 u_{cc} - 2\left(\frac{u_q}{u_c}\right) u_{cq} - \frac{\tau''}{1+\tau'}pu_q}. \tag{120}$$

We note two things here. First, we find $\varepsilon^u_{ql}\big|_{\bar{y}} = -\,\varepsilon^u_{qn}\big|_{\bar{y}}$. Second, the uncompensated and compensated elasticities of the conditional commodity demands are the same, see **?** for the formal proof.

3.2 Proof Proposition 4

Proposition 4 *The optimal non-linear marginal income tax schedule is given by the ABC-formula:*

$$\frac{T'(z_n)}{1-T'(z_n)} = \underbrace{\frac{1}{\varepsilon_{lT'}}}_{\equiv A_n}\underbrace{\frac{\int_{z_n}^{z_{\overline{n}}}(1-\lambda^*_n)\tilde{f}(z_m)\mathrm{d}z_m}{1-\tilde{F}(z_n)}}_{\equiv B_n}\underbrace{\frac{1-\tilde{F}(z_n)}{z_n\tilde{f}(z_n)}}_{\equiv C_n}, \quad \forall z_n \neq z_{\overline{n}}, z_{\underline{n}}, \tag{121}$$

The marginal cost of public funds equals one at the optimal tax system:

$$MCF \equiv \frac{\eta}{\int_{\mathcal{N}} \Psi'\lambda_n + \eta t n\frac{\partial l_n}{\partial(-T(0))} + (\eta\tau - \alpha\mu)\frac{\partial q_n}{\partial(-T(0))}f(n)\mathrm{d}n} = 1. \tag{122}$$

The optimal non-linear marginal pollution tax is given by:

$$\left(\frac{\tau'(q_n) - \frac{\alpha\mu}{\eta}}{1+\tau'(q_n)}\right)\varepsilon_{q\tau'} = -\frac{T'(z_n)}{1-T'(z_n)}\varepsilon_{lT'}\frac{\varepsilon_{lq}|_{\bar{y}}}{\varepsilon^u_{zn}}, \quad \forall z_n. \tag{123}$$

Optimal provision of environmental quality satisfies:

$$N\int_{\mathcal{N}}\frac{u_E}{u_c}\left(1+\Delta_n\right)f(n)\mathrm{d}n = \frac{\mu}{\eta}, \tag{124}$$

where

$$\Delta_n \equiv \frac{T'(z_n)}{1-T'(z_n)}\frac{\varepsilon_{lT'}}{\varepsilon^u_{zn}}\frac{\partial\ln(u_E/u_c)}{\partial\ln l_n}. \tag{125}$$

The proof proceeds in a number of steps. First, we set up the maximization problem and derive the first-order conditions. Second, we will manipulate each first-order condition using the elasticities derived in Lemma 1. Third, we collect all rewritten first-order conditions to establish the various parts of the Proposition.

3.2.1 First-order conditions

We can invert the utility function to write c_n as a function $c(q_n, l_n, u_n, E)$ of the allocation:

$$c_n = c(q_n, l_n, u_n, E), \quad \frac{\partial c_n}{\partial q_n} = -\frac{u_q}{u_c}, \quad \frac{\partial c_n}{\partial l_n} = \frac{-u_l}{u_c}, \quad \frac{\partial c_n}{\partial u_n} = \frac{1}{u_c}, \quad \frac{\partial c_n}{\partial E} = -\frac{u_E}{u_c}, \tag{126}$$

where the derivatives are found using the implicit function theorem.

The government thus solves the following maximization problem:

$$\max_{\{l_n,q_n,u_n,E\}} N\int_{\mathcal{N}}\Psi(u_n)\mathrm{d}F(n), \tag{127}$$

$$\text{s.t. } N\int_{\mathcal{N}}\left(nl_n - c(q_n,l_n,u_n,E) - q_n\right)\mathrm{d}F(n) = R, \tag{128}$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}n} = -\frac{l_n u_l(c(q_n,l_n,u_n,E),q_n,l_n,E)}{n}. \tag{129}$$

After integrating the incentive-compatibility constraint (129) by parts, the Lagrangian for maximizing social welfare can be formulated as

$$\max_{\{l_n,q_n,u_n,E\}} \mathcal{L} \equiv \int_{\mathcal{N}} \left(\Psi(u_n) + \eta\left(nl_n - c(q_n,l_n,u_n,E) - q_n - \frac{R}{N}\right)\right) f(n)\mathrm{d}n \tag{130}$$
$$- \mu\left(\frac{E-E_0}{N} + \alpha\int_{\mathcal{N}} q_n \mathrm{d}F(n)\right)$$
$$+ \int_{\mathcal{N}} \left(\theta_n \frac{l_n u_l(c(q_n,l_n,u_n,E),q_n,l_n,E)}{n} - u_n\frac{\mathrm{d}\theta_n}{\mathrm{d}n}\right)\mathrm{d}n + \theta_{\overline{n}}u_{\overline{n}} - \theta_{\underline{n}}u_{\underline{n}},$$

where θ_n is the Lagrangian multiplier associated with the differential equation for utility (129). η is the Lagrangian multiplier on the economy's resource constraint and μ is the multiplier on the environmental technology. The first-order conditions with respect to l_n, q_n, u_n, and E are given by:

$$\frac{\partial\mathcal{L}}{\partial l_n} = \eta\left(n - \frac{\partial c_n}{\partial l_n}\right)f(n) + \frac{\theta_n u_l}{n}\left(1 + \frac{l_n u_{ll}}{u_l} + \frac{l_n u_{lc}}{u_l}\frac{\partial c_n}{\partial l_n}\right) = 0, \quad \forall n, \tag{131}$$

$$\frac{\partial\mathcal{L}}{\partial q_n} = -\eta\left(1 + \frac{\partial c_n}{\partial q_n}\right)f(n) - \mu\alpha f(n) + \frac{\theta_n l_n}{n}\left(u_{lc}\frac{\partial c_n}{\partial q_n} + u_{lq}\right) = 0, \quad \forall n, \tag{132}$$

$$\frac{\partial\mathcal{L}}{\partial u_n} = \left(\Psi' - \eta\frac{\partial c_n}{\partial u_n}\right)f(n) + \frac{\theta_n u_l}{n}\frac{l_n u_{lc}}{u_l}\frac{\partial c_n}{\partial u_n} - \frac{\mathrm{d}\theta_n}{\mathrm{d}n} = 0, \quad \forall n \neq \overline{n}, \underline{n}, \tag{133}$$

$$\frac{\partial\mathcal{L}}{\partial u_{\underline{n}}} = -\theta_{\underline{n}} = 0, \quad \frac{\partial\mathcal{L}}{\partial u_{\overline{n}}} = \theta_{\overline{n}} = 0, \tag{134}$$

$$\frac{\partial\mathcal{L}}{\partial E} = \int_{\mathcal{N}}\left[\left(-\frac{\mu}{N} - \eta\frac{\partial c_n}{\partial E}\right)f(n) + \frac{\theta_n l_n}{n}\left(u_{lE} + u_{lc}\frac{\partial c_n}{\partial E}\right)\right]\mathrm{d}n = 0. \tag{135}$$

These first-order conditions are rewritten in an number of steps.

3.2.2 Rewriting the first-order condition for l_n

By substituting $\partial c_n/\partial l_n = -u_l/u_c = (1-T')n$ from (126), the first-order condition for l_n (131) can be written as:

$$\frac{T'(z_n)}{1-T'(z_n)} = \frac{u_c\theta_n/\eta}{nf(n)}\left(1 + \frac{l_n u_{ll}}{u_l} - \frac{l_n u_{lc}}{u_c}\right). \tag{136}$$

Next, use the defintions for the elasticities ε^u_{ln}, $\varepsilon_{lT'}$, ε^u_{zn} and $\varepsilon_{zT'}$ from (88), (92), (89) and (93) to derive that the elasticity term in (136) equals:

$$1 + \frac{l_n u_{ll}}{u_l} - \frac{l_n u_{lc}}{u_c} = \frac{1+\varepsilon^u_{ln}}{\varepsilon_{lT'}} = \frac{\varepsilon^u_{zn}}{\varepsilon_{zT'}}. \tag{137}$$

Using (137) in (136) then gives the following formula for the income tax:

$$\frac{T'(z_n)}{1-T'(z_n)} = \frac{u_c\theta_n/\eta}{nf(n)}\frac{(1+\varepsilon^u_{ln})}{\varepsilon_{lT'}} = \frac{u_c\theta_n/\eta}{nf(n)}\frac{\varepsilon^u_{zn}}{\varepsilon_{zT'}}. \tag{138}$$

3.2.3 Rewriting the first-order condition for q_n

By substituting $\partial c_n/\partial q_n = -u_q/u_c = -(1+\tau'(q_n))$ from (126), the first-order condition for q_n can be rewritten as:

$$\frac{\tau'(q_n) - \frac{\alpha\mu}{\eta}}{1+\tau'(q_n)} = \frac{u_c\theta_n/\eta}{nf(n)}\frac{l_n}{u_q}\left(\frac{u_q}{u_c}u_{lc} - u_{lq}\right). \tag{139}$$

Next, combine the defintions for $\varepsilon_{ql}|_{\bar{y}}$ from (118) and $\varepsilon_{q\tau'}$ from (111) to find:

$$\frac{l_n}{u_q}\left(\frac{u_q}{u_c}u_{cl} - u_{ql}\right) = -\frac{\varepsilon^u_{ql}\big|_{\bar{y}}}{\varepsilon_{q\tau'}}. \tag{140}$$

Hence, substituting (140) in (139) gives the following expression for the pollution tax rate:

$$\frac{\tau'(q_n) - \frac{\alpha\mu}{\eta}}{1+\tau'(q_n)} = -\frac{u_c\theta_n/\eta}{nf(n)}\frac{\varepsilon^u_{ql}\big|_{\bar{y}}}{\varepsilon_{q\tau'}}. \tag{141}$$

Use (138) in (141) to find the optimal non-linear pollution tax in the Proposition:

$$\left(\frac{\tau'(q_n) - \frac{\alpha\mu}{\eta}}{1+\tau'(q_n)}\right)\varepsilon_{q\tau'} = -\frac{T'(z_n)}{1-T'(z_n)}\varepsilon_{zT'}\frac{\varepsilon^u_{lq}\big|_{\bar{y}}}{\varepsilon^u_{zn}}. \tag{142}$$

3.2.4 Solution for $u_c\theta_n/\eta$

We will solve for $u_c\theta_n/\eta$, which enters the optimal income tax on the right-hand side of (138). First, rewrite the first-order condition for the level of utility u_n (133) using $\partial c_n/\partial u_n = 1/u_c$ from (126) to find:

$$\frac{u_c}{\eta}\frac{\mathrm{d}\theta_n}{\mathrm{d}n} = \left(\frac{\Psi' u_c}{\eta} - 1\right) f(n) + \frac{u_c\theta_n/\eta}{n}\frac{l_n u_{lc}}{u_c}. \tag{143}$$

We define a composite multiplier Θ_n, and substitute $l_n \equiv z_n/n$:

$$\Theta_n \equiv \frac{u_c(c_n, q_n, l_n, E)\theta_n}{\eta} = \frac{u_c(c_n, q_n, z_n/n, E)\theta_n}{\eta}. \tag{144}$$

Θ_n has total derivative:

$$\frac{\mathrm{d}\Theta_n}{\mathrm{d}n} = \frac{\mathrm{d}\theta_n}{\mathrm{d}n}\frac{u_c}{\eta} - \frac{\theta_n}{\eta}\frac{u_{cl}l_n}{n} + \frac{\theta_n}{\eta}u_{cc}\frac{\mathrm{d}c_n}{\mathrm{d}n} + \frac{\theta_n}{\eta}\frac{u_{cl}}{n}\frac{\mathrm{d}z_n}{\mathrm{d}n} + \frac{\theta_n}{\eta}u_{cq}\frac{\mathrm{d}q_n}{\mathrm{d}n} + \frac{\theta_n}{\eta}u_{cE}\frac{\mathrm{d}E}{\mathrm{d}n}. \tag{145}$$

Note that $\frac{\mathrm{d}E}{\mathrm{d}n} = 0$ since environmental quality is the same for everyone.

Totally differentiate the household budget constraint (77) and use the first-order conditons (78) to find:

$$\frac{\mathrm{d}c_n}{\mathrm{d}n} = (1 - T'(z_n))\frac{\mathrm{d}z_n}{\mathrm{d}n} - (1+\tau')\frac{\mathrm{d}q_n}{\mathrm{d}n} = \frac{-u_l}{nu_c}\frac{\mathrm{d}z_n}{\mathrm{d}n} - \frac{u_q}{u_c}\frac{\mathrm{d}q_n}{\mathrm{d}n}. \tag{146}$$

Therefore, substitution of (146) in (145) yields:

$$\frac{\mathrm{d}\Theta_n}{\mathrm{d}n} = \frac{\mathrm{d}\theta_n}{\mathrm{d}n}\frac{u_c}{\eta} - \frac{\theta_n}{\eta}\frac{u_{cl}l_n}{n} + \frac{\theta_n}{n\eta}\left(u_{cl} - \frac{u_{cc}u_l}{u_c}\right)\frac{\mathrm{d}z_n}{\mathrm{d}n} + \frac{\theta_n}{\eta}\left(\frac{u_{cc}u_q}{u_c} - u_{cq}\right)\frac{\mathrm{d}q_n}{\mathrm{d}n}. \tag{147}$$

Next, use the income elasticity of labor supply ε_l^I in (90) and the tax elasticity of labor supply $\varepsilon_{lT'}$ in (92) to find an expression for $\left(u_{cl} - \frac{u_{cc}u_l}{u_c}\right)$:

$$u_{lc} - \frac{u_l u_{cc}}{u_c} = \frac{\varepsilon_l^I}{n(1-T')}\frac{-u_l}{l\varepsilon_{lT'}}. \tag{148}$$

Similarly, use the income elasticity for commodity demands (109), the tax elasticity of commodity demands $\varepsilon_{q\tau'}$ (111), and use $\frac{u_q}{u_c} = 1 + \tau'$ to derive an expression for $(\frac{u_{cc}u_q}{u_c} - u_{cq})$:

$$\frac{u_q}{u_c}u_{cc} - u_{qc} = -\frac{\varepsilon_q^I}{(1+\tau')q_n}\frac{u_q}{\varepsilon_{q\tau'}}. \tag{149}$$

Thus, substituting (148) and (149) into (147) results in:

$$\frac{\mathrm{d}\Theta_n}{\mathrm{d}n} = \frac{\mathrm{d}\theta_n}{\mathrm{d}n}\frac{u_c}{\eta} - \frac{\theta_n}{\eta}\frac{u_{cl}l_n}{n} + \frac{\varepsilon_l^I}{(1-T')}\frac{-u_l}{n}\frac{\theta_n/\eta}{n}\frac{\varepsilon_{zn}^u}{\varepsilon_{lT'}} - \frac{\theta_n}{n\eta}\frac{\varepsilon_q^I}{(1+\tau')}u_q\frac{\varepsilon_{qn}^u}{\varepsilon_{q\tau'}}. \tag{150}$$

Use the first-order condition for l_n (138) to derive:

$$T'(z_n)f(n) = \frac{-u_l}{n}\frac{\theta_n}{n\eta}\frac{\varepsilon_{zn}^u}{\varepsilon_{lT'}}. \tag{151}$$

And, use $\varepsilon_{qn}^u\big|_{\bar{y}} = -\left.\varepsilon_{ql}^u\right|_{\bar{y}}$ in the first-order condition for q_n (141) to derive:

$$\left(\tau'(q_n) - \frac{\alpha\mu}{\eta}\right) f(n) = \frac{\theta_n/\eta}{n}u_q\frac{\varepsilon_{qn}^u}{\varepsilon_{q\tau'}}. \tag{152}$$

Note that $\varepsilon_{qn}^u\big|_{\bar{y}} = \varepsilon_{qn}$ because the first-order condition for q_n is evaluated for a given level of net income y_n, since the latter is pinned down by the first-order condition for l_n.

Substitution of (151) and (152) in (150) gives:

$$\frac{\mathrm{d}\Theta_n}{\mathrm{d}n} = \frac{\mathrm{d}\theta_n}{\mathrm{d}n}\frac{u_c}{\eta} - \frac{\theta_n}{\eta}\frac{u_{cl}l_n}{n} + \frac{\varepsilon_l^I}{(1-T')}T'f(n) + \frac{\varepsilon_q^I}{(1+\tau')}\left(\tau' - \frac{\alpha\mu}{\eta}\right)f(n). \tag{153}$$

And, substution of the income elasticities from (90) and (109) in (153) gives:

$$\frac{\mathrm{d}\Theta_n}{\mathrm{d}n} = \frac{\mathrm{d}\theta_n}{\mathrm{d}n}\frac{u_c}{\eta} - \frac{\theta_n}{\eta}\frac{u_{cl}l_n}{n} + nT'(z_n)\frac{\partial l_n}{\partial\rho}f(n) + \left(\tau'(q_n) - \frac{\alpha\mu}{\eta}\right)\frac{\partial q_n}{\partial\rho}f(n). \tag{154}$$

By combining (143) and (154) we can rewrite the first-order condition for u_n as

$$\frac{\mathrm{d}\Theta_n}{\mathrm{d}n} = \left(\frac{\Psi'(u_n)u_c}{\eta} + nT'(z_n)\frac{\partial l_n}{\partial \rho} f(n) + \left(\tau'(q_n) - \frac{\alpha\mu}{\eta} \right) \frac{\partial q_n}{\partial \rho} f(n) - 1 \right) f(n). \tag{155}$$

Integrating (155), using a transversality condition from (134) yields the solution for Θ_n:

$$\Theta_n = \frac{u_c \theta_n}{\eta} = \int_n^{\overline{n}} (1 - \lambda_m^*) f(m)\mathrm{d}m, \tag{156}$$

where

$$\lambda_m^* \equiv \frac{\Psi'(u_m)u_c}{\eta} + mT'(z_m)\frac{\partial l_m}{\partial(-T(0))} + \left(\tau'(q_m) - \frac{\alpha\mu}{\eta} \right) \frac{\partial q_m}{\partial(-T(0))}. \tag{157}$$

Note that the income effects of an increase in the intercept $-T(0)$ is the same as the income effect of an increase in exogenous income ρ. Consequently, substituting (156) in (138) gives the optimal tax expression:

$$\frac{T'(z_n)}{1 - T'(z_n)} = \frac{1}{\varepsilon_{lT'}} \frac{\int_n^{\overline{n}} (1 - \lambda_m^*) f(m)\mathrm{d}m}{1 - F(n)} \frac{(1 - F(n))\varepsilon_{zn}^u}{nf(n)}. \tag{158}$$

Finally, use the trick by **?** to write the optimal tax formula in terms of earnings densities making use of the fact that $\tilde{F}(z_n) \equiv F(n)$ so that

$$\tilde{f}(z_n) z_n \varepsilon_{zn}^u = nf(n), \tag{159}$$

$$1 - F(n) = \int_n^{\overline{n}} f(n)\mathrm{d}n = \int_{z_n}^{z_{\overline{n}}} \tilde{f}(z_n)\mathrm{d}z_n = 1 - \tilde{F}(z_n). \tag{160}$$

Hence, using (159) and (160) in (158) yields the optimal income tax as in the Proposition:

$$\frac{T'(z_n)}{1 - T'(z_n)} = \frac{1}{\varepsilon_{lT'}} \frac{\int_{z_n}^{z_{\overline{n}}} (1 - \lambda_m^*) \tilde{f}(z_m)\mathrm{d}z_m}{1 - \tilde{F}(z_n)} \frac{(1 - \tilde{F}(z_n))}{\tilde{z}_n \tilde{f}(z_n)}. \tag{161}$$

3.2.5 Marginal cost of public funds

Use the first transversality condition (134) to solve for $\Theta_{\underline{n}} = 0$:

$$\Theta_{\underline{n}} = \frac{u_c \theta_{\underline{n}}}{\eta} = \int_{\underline{n}}^{\overline{n}} (1 - \lambda_m^*) f(m)\mathrm{d}m = 0. \tag{162}$$

Hence, using the definition of λ_n^* from (157), we find the marginal cost of funds as in the Proposition:

$$MCF \equiv \frac{\eta}{\int_{\mathcal{N}} \Psi'\lambda_n + \eta t n \frac{\partial l_n}{\partial(-T(0))} + (\eta\tau - \alpha\mu)\frac{\partial q_n}{\partial(-T(0))} f(n)\mathrm{d}n} = 1. \tag{163}$$

3.2.6 Optimal provision of environmental quality

Use $\frac{\partial c_n}{\partial E} = -\frac{u_E}{u_c}$ in first-order condition (135) and rewrite:

$$N\int_{\mathcal{N}} \frac{u_E}{u_c} \left[1 + \frac{u_c\theta_n/\eta}{nf(n)} \left(\frac{l_n u_{lE}}{u_E} - \frac{l_n u_{lc}}{u_c} \right) \right] f(n)\mathrm{d}n = \frac{\mu}{\eta}. \tag{164}$$

Define

$$\Delta_n \equiv \frac{u_c\theta_n/\eta}{nf(n)} \left(\frac{l_n u_{lE}}{u_E} - \frac{l_n u_{lc}}{u_c} \right). \tag{165}$$

Use the optimal income tax (138), and note that $\frac{l_n u_{lE}}{u_E} - \frac{l_n u_{lc}}{u_c} = \frac{l_n}{u_E}\frac{\partial u_E}{\partial l_n} - \frac{l_n}{u_c}\frac{\partial u_c}{\partial l_n} = \frac{\partial \ln(u_E/u_c)}{\partial \ln l_n}$ to find

$$\Delta_n = \frac{T'(z_n)}{1 - T'(z_n)} \frac{\varepsilon_{zT'}}{\varepsilon_{zn}^u} \left(\frac{\partial \ln(u_E/u_c)}{\partial \ln l_n} \right) \tag{166}$$

Consequently, substituting (166) in (164) yields optimal environmental quality as in the Proposition:

$$N\int_{\mathcal{N}} \frac{u_E}{u_c} (1 + \Delta_n) f(n)\mathrm{d}n = \frac{\mu}{\eta}. \tag{167}$$